Indirect controllability of some linear parabolic systems of m equations with m − 1 controls involving coupling terms of zero or first order

Michel Duprez, Pierre Lissy

Indirect controllability of some linear parabolic systems of m equations with $m-1$ controls involving coupling terms of zero or first order.

Michel Duprez[*], Pierre Lissy[†]

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Abstract

This paper is devoted to the study of the null and approximate controllability for some classes of linear coupled parabolic systems with less controls than equations. More precisely, for a given bounded domain Ω in $\mathbb{R}^N$ ($N \in \mathbb{N}^*$), we consider a system of m linear parabolic equations ($m \geqslant 2$) with coupling terms of first and zero order, and $m-1$ controls localized in some arbitrary nonempty open subset ω of Ω. In the case of constant coupling coefficients, we provide a necessary and sufficient condition to obtain the null or approximate controllability in arbitrary small time. In the case $m = 2$ and $N = 1$, we also give a generic sufficient condition to obtain the null or approximate controllability in arbitrary small time for general coefficients depending on the space and times variables, provided that the supports of the coupling terms intersect the control domain ω. The results are obtained thanks to the fictitious control method together with an algebraic method and some appropriate Carleman estimates.

1 Introduction

1.1 Presentation of the problem and main results

Let $T > 0$, let Ω be a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}^*$) of class $\mathcal{C}^2$ and let ω be an arbitrary nonempty open subset of Ω. Let $Q_T := (0,T)\times\Omega$, $q_T := (0,T)\times\omega$, $\Sigma_T := (0,T)\times\partial\Omega$ and $m \geqslant 2$. We consider the following system of m parabolic linear equations, where the $m-1$ first equations are controlled:

$$\left\{\begin{array}{ll}
\partial_t y_1 = \operatorname{div}(d_1\nabla y_1) + \sum_{i=1}^m g_{1i}\cdot\nabla y_i + \sum_{i=1}^m a_{1i} y_i + \mathbb{1}_\omega u_1 & \text{in } Q_T,\\
\partial_t y_2 = \operatorname{div}(d_2\nabla y_2) + \sum_{i=1}^m g_{2i}\cdot\nabla y_i + \sum_{i=1}^m a_{2i} y_i + \mathbb{1}_\omega u_2 & \text{in } Q_T,\\
\vdots & \\
\partial_t y_{m-1} = \operatorname{div}(d_{m-1}\nabla y_{m-1}) + \sum_{i=1}^m g_{1(m-1)i}\cdot\nabla y_i + \sum_{i=1}^m a_{(m-1)i} y_i + \mathbb{1}_\omega u_{m-1} & \text{in } Q_T,\\
\partial_t y_m = \operatorname{div}(d_m\nabla y_m) + \sum_{i=1}^m g_{mi}\cdot\nabla y_i + \sum_{i=1}^m a_{mi} y_i & \text{in } Q_T,\\
y_1 = \ldots = y_m = 0 & \text{on } \Sigma_T,\\
y_1(0,\cdot) = y_1^0, \ldots,\ y_m(0,\cdot) = y_m^0 & \text{in } \Omega,
\end{array}\right. \tag{1.1}$$

where $y^0 := (y_1^0, \ldots, y_m^0) \in L^2(\Omega)^m$ is the initial condition and $u := (u_1, \ldots, u_{m-1}) \in L^2(Q_T)^{m-1}$ is the control. The zero and first order coupling terms $(a_{ij})_{1\leqslant i,j\leqslant m}$ and $(g_{ij})_{1\leqslant i,j\leqslant m}$ are assumed to

[*]Laboratoire de Mathématiques de Besançon UMR CNRS 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon Cedex, France, michel.duprez@univ-fcomte.fr,

[†]CEREMADE, UMR 7534, Université Paris-Dauphine & CNRS, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France, lissy@ceremade.dauphine.fr.

be respectively in $L^\infty(Q_T)$ and in $L^\infty(0,T;W^1_\infty(\Omega)^N)$. Given some $l\in\{1,...,m\}$, the second order elliptic self-adjoint operator $\mathrm{div}(d_l\nabla)$ is given by

$$\mathrm{div}(d_l\nabla)=\sum_{i,j=1}^m \partial_i(d_l^{ij}\partial_j),$$

with

$$\begin{cases} d_l^{ij}\in W^1_\infty(Q_T),\\ d_l^{ij}=d_l^{ji} \text{ in } Q_T,\end{cases}$$

where the coefficients d_l^{ij} satisfy the uniform ellipticity condition

$$\sum_{i,j=1}^N d_l^{ij}\xi_i\xi_j\geqslant d_0|\xi|^2 \text{ in } Q_T,\ \forall\xi\in\mathbb{R}^N,$$

for a constant $d_0>0$.

In order to simplify the notation, from now on, we will denote by

$$D:=\mathrm{diag}(d_1,\dots,d_m),\ \ G:=(g_{ij})_{1\leqslant i,j\leqslant m}\in\mathcal{M}_m(\mathbb{R}^N),$$
$$A:=(a_{ij})_{1\leqslant i,j\leqslant m}\in\mathcal{M}_m(\mathbb{R}),\ B:=\begin{pmatrix}\mathrm{diag}(1,\dots,1)\\ 0\end{pmatrix}\in\mathcal{M}_{m,m-1}(\mathbb{R}),$$

so that we can write System (1.1) as

$$\begin{cases}\partial_t y=\mathrm{div}(D\nabla y)+G\cdot\nabla y+Ay+\mathbb{1}_\omega Bu & \text{in } Q_T,\\ y=0 & \text{on } \Sigma_T,\\ y(0,\cdot)=y^0 & \text{in } \ \Omega.\end{cases}\tag{1.2}$$

It is well-known (see for instance [16, Th. 3 & 4, p. 356-358]) that for any initial data $y^0\in L^2(\Omega)^m$ and $u\in L^2(0,T;H^{-1}(\Omega))^{m-1}$, System (1.2) admits a unique solution y in $W(0,T)^m$, where

$$W(0,T):=L^2(0,T;H^1_0(\Omega))\cap H^1(0,T;H^{-1}(\Omega))\hookrightarrow\mathcal{C}^0([0,T];L^2(\Omega)).\tag{1.3}$$

Moreover, one can prove (see for instance [16, Th. 5, p. 360]) that if $y^0\in H^1_0(\Omega)^m$ and $u\in L^2(Q_T)^{m-1}$, then the solution y is in $W^{2,1}_2(Q_T)^m$, where

$$W^{2,1}_2(Q_T):=L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))\cap H^1(0,T;L^2(\Omega))\hookrightarrow\mathcal{C}^0([0,T];H^1_0(\Omega)).\tag{1.4}$$

The main goal of this article is to analyse the null controllability and approximate controllability of System (1.1). Let us recall the definition of these notions. It will be said that

- System (1.1) is *null controllable* at time T if for every initial condition $y^0\in L^2(\Omega)^m$, there exists a control $u\in L^2(Q_T)^{m-1}$ such that the solution y in $W(0,T)^m$ to System (1.1) satisfies

$$y(T)\equiv 0 \ \text{ in } \ \Omega.$$

- System (1.1) is *approximately controllable* at time T if for every $\varepsilon>0$, every initial condition $y^0\in L^2(\Omega)^m$ and every $y_T\in L^2(\Omega)^m$, there exists a control $u\in L^2(Q_T)^{m-1}$ such that the solution y in $W(0,T)^m$ to System (1.1) satisfies

$$\|y(T)-y_T\|^2_{L^2(\Omega)^m}\leqslant\varepsilon.$$

Let us remark that if System (1.1) is null controllable on the time interval $(0,T)$, then it is also approximately controllable on the time interval $(0,T)$ (this is an easy consequence of usual results of backward uniqueness concerning parabolic equations as given for example in [10]).

Our first result gives a necessary and sufficient condition for null (or approximate) controllability of System (1.1) in the case of constant coefficients. This condition is the natural one that can be expected, since it only means that the last equation is coupled with at least one of the others.

Theorem 1. *Let us assume that D, G and A are* **constant in space and time**. *Then System* (1.1) *is null (resp. approximately) controllable at time $T>0$ if and only if there exists $i_0 \in \{1,...,m-1\}$ such that*

$$g_{mi_0} \neq 0 \quad \text{or} \quad a_{mi_0} \neq 0. \tag{1.5}$$

Our second result concerns the case of general coefficients depending on space and time variables, in the particular case of two equations and one space dimension (i.e. $m=2$ and $N=1$), and gives a controllability result under some technical conditions on the coefficients (see (1.7) and (1.8)) coming from the algebraic solvability (see Section 3.1). To understand why this kind of condition appears here, we refer to the simple example given in [26, Ex. 1, Sec. 1.3]. Let us emphasize that Condition (1.8), which is clearly technical since it does not even cover the case of constant coefficients, is *generic* as soon as we restrict to the coupling coefficients verifying $g_{21} \neq 0$ on $(0,T)\times\omega$, in the following sense: it only requires some regularity on the coefficients and a given determinant, involving some coefficients and their derivatives, to be nonzero. Since this condition may seem a little bit intricate, we will give in Remark 1 some particular examples that clarify the scope of Theorem 2.

Theorem 2. *Let us assume that $m=2$, $N=1$, $\Omega=(0,L)$ with $L>0$, and consider the following system:*

$$\begin{cases} \partial_t y_1 = \partial_x(d_1\partial_x y_1) + g_{11}\partial_x y_1 + g_{12}\partial_x y_2 + a_{11}y_1 + a_{12}y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \partial_x(d_2\partial_x y_2) + g_{21}\partial_x y_1 + g_{22}\partial_x y_2 + a_{21}y_1 + a_{22}y_2 & \text{in } Q_T, \\ y_1(\cdot,0) = y_2(\cdot,0) = y_1(\cdot,L) = y_2(\cdot,L) = 0 & \text{on } (0,T), \\ y_1(0,\cdot) = y_1^0, \ y_2(0,\cdot) = y_2^0, & \text{in } (0,L), \end{cases} \tag{1.6}$$

where $y^0 = (y_1^0, y_2^0) \in L^2(0,L)^2$ is the initial condition.

Then System (1.1) *is null (resp. approximately) controllable at time T if there exists an open subset $(a,b)\times\mathcal{O} \subseteq q_T$ where one of the following conditions is verified:*

(i) Coefficients of System (1.6) *satisfy $d_i, \ g_{ij}, \ a_{ij} \in \mathcal{C}^1((a,b),\mathcal{C}^2(\mathcal{O}))$ for $i,j=1,2$ and*

$$\begin{cases} g_{21} = 0 \text{ and } a_{21} \neq 0 & \text{in } (a,b)\times\mathcal{O}, \\ 1/a_{21} \in L^\infty(\mathcal{O}) & \text{in } (a,b). \end{cases} \tag{1.7}$$

(ii) Coefficients of System (1.6) *satisfy $d_i, \ g_{ij}, \ a_{ij} \in \mathcal{C}^3((a,b),\mathcal{C}^7(\mathcal{O}))$ for $i=1,2$ and*

$$|\det(H(t,x))| > C \textit{ for every } (t,x) \in (a,b)\times\mathcal{O}, \tag{1.8}$$

where

$$H := \begin{pmatrix} -a_{21}+\partial_x g_{21} & g_{21} & 0 & 0 & 0 & 0 \\ -\partial_x a_{21}+\partial_{xx} g_{21} & -a_{21}+2\partial_x g_{21} & 0 & g_{21} & 0 & 0 \\ -\partial_t a_{21}+\partial_{tx} g_{21} & \partial_t g_{21} & -a_{21}+\partial_x g_{21} & 0 & g_{21} & 0 \\ -\partial_{xx} a_{21}+\partial_{xxx} g_{21} & -2\partial_x a_{21}+3\partial_{xx} g_{21} & 0 & -a_{21}+3\partial_x g_{21} & 0 & g_{21} \\ -a_{22}+\partial_x g_{22} & g_{22}-\partial_x d_2 & -1 & -d_2 & 0 & 0 \\ -\partial_x a_{22}+\partial_{xx} g_{22} & -a_{22}+2\partial_x g_{22}-\partial_{xx} d_2 & 0 & g_{22}-2\partial_x d_2 & -1 & -d_2 \end{pmatrix}. \tag{1.9}$$

Remark 1. (a) We will see during the proof that, in Item (ii) of Theorem 2, taking into account the derivatives of the appearing in (1.9), (3.5) and (3.9), and the regularity needed for the control in Proposition 3.4, we need only the following regularity for the coefficients:

$$d_i \in \mathcal{C}^1((a,b),\mathcal{C}^2(\mathcal{O})), \ g_{ij} \in \mathcal{C}^0((a,b),\mathcal{C}^2(\mathcal{O})), \ a_{ij} \in \mathcal{C}^0((a,b),\mathcal{C}^1(\mathcal{O})) \tag{1.10}$$

for all $i,j\in\{1,2\}$ in the first case (i) and

$$\left\{\begin{array}{l}
d_1\in\mathcal{C}^1((a,b),\mathcal{C}^4(\mathcal{O}))\cap\mathcal{C}^2((a,b),\mathcal{C}^1(\mathcal{O})),\\
g_{11},\ g_{12}\in\mathcal{C}^0((a,b),\mathcal{C}^4(\mathcal{O}))\cap\mathcal{C}^1((a,b),\mathcal{C}^1(\mathcal{O})),\\
a_{11},\ a_{12}\in\mathcal{C}^0((a,b),\mathcal{C}^3(\mathcal{O}))\cap\mathcal{C}^1((a,b),\mathcal{C}^0(\mathcal{O})),\\
g_{21}\in\mathcal{C}^0((a,b),\mathcal{C}^7(\mathcal{O}))\cap\mathcal{C}^3((a,b),\mathcal{C}^1(\mathcal{O})),\\
d_2,\ g_{22}\in\mathcal{C}^0((a,b),\mathcal{C}^6(\mathcal{O}))\cap\mathcal{C}^2((a,b),\mathcal{C}^2(\mathcal{O})),\\
a_{22}\in\mathcal{C}^0((a,b),\mathcal{C}^5(\mathcal{O}))\cap\mathcal{C}^2((a,b),\mathcal{C}^1(\mathcal{O})),\\
a_{21}\in\mathcal{C}^0((a,b),\mathcal{C}^6(\mathcal{O}))\cap\mathcal{C}^3((a,b),\mathcal{C}^0(\mathcal{O})),
\end{array}\right. \tag{1.11}$$

in the second case (ii).

(b) One can easily compute explicitly the determinant of matrix H appearing in (1.8):

$$\begin{array}{l}
\det(H)=2\frac{\partial a_{21}}{\partial x}\frac{\partial d_2}{\partial x}g_{21}^2-4\frac{\partial a_{21}}{\partial x}d_2\frac{\partial g_{21}}{\partial x}g_{21}+\frac{\partial^2 a_{21}}{\partial x^2}d_2g_{21}^2+2a_{21}\frac{\partial a_{21}}{\partial x}d_2g_{21}-\frac{\partial a_{21}}{\partial x}g_{21}^2g_{22}+\frac{\partial a_{21}}{\partial t}g_{21}^2\\
-4a_{21}\frac{\partial d_2}{\partial x}\frac{\partial g_{21}}{\partial x}g_{21}+a_{21}\frac{\partial^2 d_2}{\partial x^2}g_{21}^2+a_{21}^2\frac{\partial d_2}{\partial x}g_{21}-3a_{21}d_2\frac{\partial^2 g_{21}}{\partial x^2}g_{21}+6a_{21}d_2(\frac{\partial g_{21}}{\partial x})^2-2a_{21}^2d_2\frac{\partial g_{21}}{\partial x}+a_{21}\frac{\partial g_{21}}{\partial x}g_{21}g_{22}\\
-a_{21}\frac{\partial g_{21}}{\partial t}g_{21}-a_{21}g_{21}^2\frac{\partial g_{22}}{\partial x}+\frac{\partial a_{22}}{\partial x}g_{21}^3-\frac{\partial^2 d_2}{\partial x^2}\frac{\partial g_{21}}{\partial x}g_{21}^2-2\frac{\partial d_2}{\partial x}\frac{\partial^2 g_{21}}{\partial x^2}g_{21}^2+3\frac{\partial d_2}{\partial x}(\frac{\partial g_{21}}{\partial x})^2g_{21}-d_2\frac{\partial^3 g_{21}}{\partial x^3}g_{21}^2\\
+5d_2\frac{\partial g_{21}}{\partial x}\frac{\partial^2 g_{21}}{\partial x^2}g_{21}-4d_2(\frac{\partial g_{21}}{\partial x})^3+\frac{\partial g_{21}}{\partial x}g_{21}^2\frac{\partial g_{22}}{\partial x}+\frac{\partial g_{21}}{\partial x^2}g_{21}^2g_{22}-\frac{\partial g_{21}}{\partial x}^2g_{21}g_{22}-\frac{\partial g_{21}}{\partial x\partial t}g_{21}^2+\frac{\partial g_{21}}{\partial x}\frac{\partial g_{21}}{\partial t}g_{21}\\
-g_{21}^3\frac{\partial g_{22}}{\partial x^2}.
\end{array}$$

(c) We remark that Condition (1.8) implies in particular that

$$g_{21}\neq 0 \text{ in } (a,b)\times\mathcal{O}. \tag{1.12}$$

Our conjecture is that, as in the case of constant coefficients, either the first line of (1.7) or (1.12) is sufficient as soon as we restrict to the class of coupling terms that intersect the control region, since it is the minimal conditions one can expect (as in the case of constant coefficients, this only means that the last equation is coupled with one of the others).

(d) Even though Condition (1.8) seems complicated, it can be simplified in some cases. Indeed, for example, System (1.6) is null controllable at time T if there exists an open subset $(a,b)\times\mathcal{O}\subseteq q_T$ such that

$$\left\{\begin{array}{ll}
g_{21}\equiv\kappa\in\mathbb{R}^* & \text{in } (a,b)\times\mathcal{O},\\
a_{21}\equiv 0 & \text{in } (a,b)\times\mathcal{O},\\
\partial_x a_{22}\neq\partial_{xx}g_{22} & \text{in } (a,b)\times\mathcal{O}.
\end{array}\right.$$

Another simple situation is the case where the coefficients depend only on the time variable. In this case, it is easy to check that Condition (1.8) becomes simply: there exists an open interval (a,b) of $(0,T)$ such that

$$g_{21}(t)\partial_t a_{21}(t)\neq a_{21}(t)\partial_t g_{21}(t)\quad\text{ in } (a,b).$$

(e) In fact, it is likely that one could obtain a far more general result than the one obtained in Theorem 2. By using the same reasoning, one would be able to obtain a result of controllability for arbitrary m and N, however the generic Condition (1.8) would be far more complicated and in general impossible to write down explicitly. That is the reason why we chose to treat only the case $m=2$ and $N=1$.

This paper is organized as follows. In Section 1.2 we recall some previous results and explain precisely the scope of the present contribution. In Section 1.3 we present the main method used here, that is to say the fictitious control method together with some algebraic method. Section 2 is devoted to the proof of Theorem 1. We finish with the proof of Theorem 2 in Section 3.

1.2 State of the art

The study of what is called the *indirect* controllability for linear or nonlinear parabolic coupled systems has been an intensive subject of interest these last years. The main issue is to try to control many equations with less controls than equations (and ideally only one control if possible), with the hope that one can act *indirectly* on the equations that are not directly controlled thanks to the coupling terms. For a recent survey concerning this kind of control problems, we refer to [7]. Here, we will mainly present the results related to this work, that is to say the case of the null or approximate controllability of linear parabolic systems with distributed controls.

First of all, in the case of zero order coupling terms, a necessary and sufficient algebraic condition is proved in [5] for the controllability of parabolic systems, for constant coefficients and diffusion coefficients d_i that are equal to the identity matrices. This condition is similar to the usual algebraic *Kalman rank condition* for finite-dimensional systems. These results were then extended in [6], where a necessary and sufficient condition is given for constant coefficients but different diffusion coefficients d_i (the Laplace operator Δ can also be replaced by some general time-independent elliptic operator). Moreover, in [5], some results are obtained in the case of time-dependent coefficients under a sharp sufficient condition which is similar to the sharp *Silverman-Meadows Theorem* in the finite-dimensional case.

Concerning the case of space-varying coefficients, there is currently no general theory. In the case where the support of the coupling terms intersect the control domain, the most general result is proved in [19] for parabolic systems in cascade form with one control force (and possibly one order coupling terms). We also mention [4], where a result of null controllability is proved in the case of a system of two equations with one control force, with an application to the controllability of a nonlinear system of transport-diffusion equations. In the case where the coupling regions do not intersect the control domain, only few results are known and in general there are some technical and geometrical restrictions (see for example [1], [3] or [27]). These restrictions come from the use of the *transmutation method* that requires a controllability result on some related hyperbolic system. Let us also mention [8], where the authors consider a system of two equations in one space dimension and obtain a minimal time for null controllability, when the supports of the coupling terms do not intersect the control domain.

Concerning the case of first order coupling terms, there are also only few results. The first one is [22], where the author studies notably the case of m coupled heat equations with $m-1$ control forces and obtains the null controllability of System (1.1) at any time when the following estimate holds:

$$\|u\|_{H^1(\Omega)} \leqslant C\|g_{21}\cdot\nabla u + a_{21}u\|_{L^2(\Omega)}, \tag{1.13}$$

for all $u \in H^1_0(\Omega)$. Let us emphasize that inequality (1.13) is very restrictive, because it notably implies that g_{21} has to be nonzero on each of its components (due to the H^1-norm appearing in the left-hand side). Another case is the null controllability at any time $T > 0$ of m equations with one control force, which is studied in [19], under many assumptions: the coupling matrix G has to be upper triangular (except on the controlled equation) and many coefficients of A have to be non identically equal to zero on ω, notably, in the 2×2 case, we should have

$$\begin{array}{c} g_{21} \equiv 0 \text{ in } q_T \\ \text{and} \\ (a_{21} > a_0 \text{ in } q_T \quad \text{or} \quad a_{21} < -a_0 \text{ in } q_T), \end{array} \tag{1.14}$$

for a constant $a_0 > 0$. The last result concerning first order coupling terms is the recent work [11], where the case of 2×2 and 3×3 systems with one control force is studied under some technical assumptions. Notably, in the 2×2 case, the authors assume that

$$\left\{\begin{array}{l} \text{there exists an nonempty open subset } \gamma \text{ of } \partial\omega \cap \partial\Omega, \\ \exists x_0 \in \gamma \text{ s.t. } g_{21}(t, x_0)\cdot\nu(x_0) \neq 0 \ \text{ for all } t \in [0, T], \end{array}\right. \tag{1.15}$$

where ν represents the exterior normal unit vector to the boundary $\partial\Omega$. Under these technical restrictions on the control domain and the coupling terms, System (1.1) is null controllable at any time $T > 0$.

Here we detail how our results differ from the existing ones:

1. In the case of constant coefficients, we are able to obtain a necessary and sufficient condition in the case of m equations, $m-1$ controls and coupling terms of order 0 or 1, which is the main new result. Moreover, the diffusion coefficients can be different, we are able to treat the case of as many equations as wanted and we use an inequality similar to Condition (1.13) but with the L^2-norm in the left-hand side (see Lemma 2.4). To finish, we do not need the control domain to extend up to the boundary as in Condition (1.15). The main restriction is that all the coefficients of System (1.1) must be constant.

2. In the case $N = 1$ and $m = 2$, we are able to obtain the controllability in arbitrary small time under some generic conditions which are purely technical (at least in the second item). In Theorem 2, the geometric condition (1.15) is not necessary, which is very satisfying, moreover, in the case $g_{21} = 0$, we do not need a condition of constant sign like in (1.14) on a_{21}, and the conditions on a_{21} and g_{21} given in (1.7) have to be verified only locally in space and time (contrary to (1.14) where it is local only in space).

1.3 Strategy

The method described in this section is sometimes called *fictitious control method* and has already been used for instance in [20], [14] and [2]. One important limitation of this method is that it will never be useful to treat the case where the support of the coupling terms do not intersect the control region, because, in what we call the algebraic resolution, we have to work locally on the control region.

Roughly, the method is the following: we first control the equations with m controls (one on each equation) and we try to eliminate the control on the last equation thanks to algebraic manipulations. Let us be more precise and decompose the problem into two different steps:

Analytic problem:
Find a solution (z, v) in an appropriate space to the control problem by m controls which are regular enough and are in the range of a differential operator. More precisely, solve

$$\begin{cases} \partial_t z = \mathrm{div}(D\nabla z) + G \cdot \nabla z + Az + \mathcal{N}(\mathbb{1}_{\widetilde{\omega}} v) & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ z(0,\cdot) = y^0,\ z(T,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{1.16}$$

where $\mathcal{N}$ is some differential operator to be chosen later and $\widetilde{\omega}$ is strongly included in ω. Solving Problem (1.16) is easier than solving the null controllability at time T of System (1.1), because we control System (1.16) with a control on each equation. The important points (and somehow different from the usual method) are that:

1. The control has to be of a special form (it has to be in the range of a differential operator $\mathcal{N}$),

2. The control has to be regular enough, so that it can be differentiated a certain amount of times with respect to the space and/or time variables (see the next section about the algebraic resolution).

If we look for a control v in the weighted Sobolev space $L^2(Q_T, \rho^{-1/2})$ for some weight ρ, it is well known (see, e.g. [12, Th. 2.44, p. 56–57]) that the null controllability at time T of System (1.16) is equivalent to the following *observability inequality*:

$$\int_\Omega |\psi(0,x)|^2 dx \leqslant C_{obs} \iint_{q_T} \rho |\mathcal{N}^* \psi(t,x)|^2 \, dxdt, \tag{1.17}$$

where ψ is the solution to the dual system

$$\begin{cases} -\partial_t \psi = \operatorname{div}(D\nabla\psi) - G^* \cdot \nabla\psi + A^*\psi & \text{in } Q_T, \\ \psi = 0 & \text{on } \Sigma_T, \\ \psi(T,\cdot) = \psi^0 & \text{in } \Omega. \end{cases}$$

Inequalities like (1.17) can be proved thanks to some appropriate *Carleman estimates*. The weight ρ can be chosen to be exponentially decreasing at times $t = 0$ and $t = T$, which will be useful later. In fact, we will have to adapt the usual HUM duality method to ensure that one can find such controls.

Algebraic problem:
For $f := \mathbb{1}_\omega v$, find a pair $(\widehat{z}, \widehat{v})$ (where $\widehat{v}$ now acts only on the first $m-1$ equations) in an appropriate space satisfying the following control problem:

$$\begin{cases} \partial_t \widehat{z} = \operatorname{div}(D\nabla\widehat{z}) + G \cdot \nabla\widehat{z} + A\widehat{z} + B\widehat{v} + \mathcal{N}f & \text{in } Q_T, \\ \widehat{z} = 0 & \text{on } \Sigma_T, \\ \widehat{z}(0,\cdot) = \widehat{z}(T,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{1.18}$$

and such that the spatial support of $\widehat{v}$ is strongly included in ω. We will solve this problem using the notion of *algebraic resolvability* of differential systems, which is based on ideas coming from [21, Section 2.3.8] and was already widely used in [14] and [2]. The idea is to write System (1.18) as an *underdetermined* system in the variables $\widehat{z}$ and $\widehat{v}$ and to see $\mathcal{N}f$ as a source term, so that we can write Problem (1.18) under the abstract form

$$\mathcal{L}(\widehat{z}, \widehat{v}) = \mathcal{N}f, \tag{1.19}$$

where

$$\mathcal{L}(\widehat{z}, \widehat{v}) := \quad \partial_t \widehat{z} - \operatorname{div}(D\nabla\widehat{z}) - G \cdot \nabla\widehat{z} - A\widehat{z} - B\widehat{v}. \tag{1.20}$$

The goal will be then to find a partial differential operator $\mathcal{M}$ satisfying

$$\mathcal{L} \circ \mathcal{M} = \mathcal{N}. \tag{1.21}$$

When (1.21) is satisfied, we say that System (1.18) is *algebraically resolvable*. This exactly means that one can find a solution $(\widehat{z}, \widehat{v})$ of System (1.18) which can be written as a linear combination of some derivatives of the source term $\mathcal{N}f$. The main advantage of this method is that one can only work *locally* on q_T, because the solution depends locally on the source term and then has the same support as the source term (to obtain a solution which is defined everywhere on Q_T, one just extends it by 0). This part will be explained in more details in Sections 2.1 and 3.1.

Conclusion:
If we can solve the analytic and algebraic problems, then it is easy to check that $(y, u) := (z - \widehat{z}, -\widehat{v})$ will be a solution to System (1.1) in an appropriate space and will satisfy $y(T) \equiv 0$ in Ω (for more explanations, see [14, Prop. 1] or Sections 2.4 and 3.3 of the present paper).

2 Proof of Theorem 1

Let us remind that in this case, D, G and A are constant. In Section 2.1, 2.2 and 2.3, we will always consider some $i_0 \in \{1, ..., m-1\}$ such that

$$g_{mi_0} \neq 0 \text{ or } a_{mi_0} \neq 0. \tag{2.1}$$

We will follow the strategy described in Section 1.3, and we first begin with finding some appropriate operator $\mathcal{N}$.

2.1 Algebraic resolution

We will here explain how to choose the differential operator $\mathcal{N}$ used in the next section. We will assume from now on that all differential operators of this section are defined in $\mathcal{C}^\infty(Q_T)$. The appropriate spaces will be precised in Section 2.4. We consider $\mathcal{N}$ as the operator defined for all $f := (f_1, \dots, f_m)$ by

$$\mathcal{N}(f) := \begin{pmatrix} \mathcal{N}_1 f \\ \mathcal{N}_2 f \\ \dots \\ \mathcal{N}_m f \end{pmatrix} := \begin{pmatrix} (-g_{mi_0} \cdot \nabla - a_{mi_0}) f_1 \\ (-g_{mi_0} \cdot \nabla - a_{mi_0}) f_2 \\ \dots \\ (-g_{mi_0} \cdot \nabla - a_{mi_0}) f_m \end{pmatrix}. \tag{2.2}$$

Let us recall that the definition of $\mathcal{L}$ is given in (1.20).

As explained in Section 1.3, we want find a differential operator $\mathcal{M}$ such that

$$\mathcal{L} \circ \mathcal{M} = \mathcal{N}. \tag{2.3}$$

We have the following proposition:

Proposition 2.1. *Let $\mathcal{N}$ be defined as in (2.2). Then there exists a differential operator $\mathcal{M}$ of order 1 in time and 2 in space, with constant coefficients, such that (2.3) is verified.*

Proof of Proposition 2.1. We can remark that equality (2.3) is equivalent to

$$\mathcal{M}^* \circ \mathcal{L}^* = \mathcal{N}^*. \tag{2.4}$$

The adjoint $\mathcal{L}^*$ of the operator $\mathcal{L}$ is given for all $\varphi \in \mathcal{C}^\infty(Q_T)^m$ by

$$\mathcal{L}^*\varphi \quad := \quad \begin{pmatrix} \mathcal{L}_1^*\varphi \\ \dots \\ \mathcal{L}_{2m-1}^*\varphi \end{pmatrix} = \begin{pmatrix} -\partial_t\varphi_1 - d_1\Delta\varphi_1 + \sum_{j=1}^m \{g_{j1} \cdot \nabla\varphi_j - a_{j1}\varphi_j\} \\ \dots \\ -\partial_t\varphi_m - d_m\Delta\varphi_1 + \sum_{j=1}^m \{g_{jm} \cdot \nabla\varphi_j - a_{jm}\varphi_j\} \\ \varphi_1 \\ \dots \\ \varphi_{m-1} \end{pmatrix}. \tag{2.5}$$

Now we apply $g_{mi_0} \cdot \nabla - a_{mi_0}$ to the $(m+i)^{th}$ line for $i \in \{1, ..., m-1\}$ and we add $(\partial_t + d_{i_0}\Delta)\mathcal{L}^*_{m+i_0}\varphi + \sum_{j=1}^{m-1}(-g_{ji_0} \cdot \nabla + a_{ji_0})\mathcal{L}^*_{m+j}\varphi$ to the $(i_0)^{th}$ line. Hence, remarking that $\mathcal{L}^*_{m+i}\varphi = \varphi_i$ ($i \in \{1, ..., m-1\}$), we obtain

$$\begin{pmatrix} (g_{mi_0} \cdot \nabla - a_{mi_0})\mathcal{L}^*_{m+1}\varphi \\ \dots \\ (g_{mi_0} \cdot \nabla - a_{mi_0})\mathcal{L}^*_{2m-1}\varphi \\ \mathcal{L}^*_{i_0}\varphi + (\partial_t + d_{i_0}\Delta)\mathcal{L}^*_{m+i_0}\varphi + \sum_{j=1}^{m-1}(-g_{ji_0} \cdot \nabla + a_{ji_0})\mathcal{L}^*_{m+j}\varphi \end{pmatrix} = \mathcal{N}^*\varphi.$$

Thus if we define $\mathcal{M}^*$ for $\psi := (\psi_1, ..., \psi_{2m-1}) \in \mathcal{C}^\infty(Q_T)^{2m-1}$ by

$$\mathcal{M}^* \begin{pmatrix} \psi_1 \\ \dots \\ \psi_{2m-1} \end{pmatrix} := \begin{pmatrix} (g_{mi_0}\nabla - a_{mi_0})\psi_{m+1} \\ \dots \\ (g_{mi_0}\nabla - a_{mi_0})\psi_{2m-1} \\ \psi_{i_0} + (\partial_t + d_{i_0}\Delta)\psi_{m+i_0} + \sum_{j=1}^{m-1}(-g_{ji_0} \cdot \nabla + a_{ji_0})\psi_{m+j} \end{pmatrix}, \tag{2.6}$$

then equality (2.4) is satisfied and hence equality (2.3) also. Moreover, the coefficients of $\mathcal{M}^*$ are constant, hence it is also the case for the coefficients of $\mathcal{M}$. ∎

2.2 An appropriate Carleman estimate

Let us consider the following dual system associated to System (1.16)

$$\begin{cases} -\partial_t \psi = \operatorname{div}(D\nabla\psi) - G^* \cdot \nabla\psi + A^*\psi & \text{in } Q_T, \\ \psi = 0 & \text{on } \Sigma_T, \\ \psi(T,\cdot) = \psi^0 & \text{in } \Omega. \end{cases} \tag{2.7}$$

The two main results of this section are Propositions 2.2 and 2.3, which are respectively some Carleman estimate and observability inequality. The particularity of theses equalities is that the observation will not be directly the L^2-norm of the solution ψ to System (2.7) on the subset ω, but it will be the L^2-norm of some linear combination of ψ and its derivatives of first order on the subset ω. This particular form will be used in the next section to construct a solution to the analytic control Problem (1.16).

Let ω_0, ω_1 and ω_2 be three nonempty open subsets included in ω satisfying

$$\overline{\omega_2} \subset \omega_1, \ \overline{\omega_1} \subset \omega_0 \text{ and } \overline{\omega_0} \subset \omega.$$

Before stating the Carleman estimate, let us introduce some notations. For $s, \lambda > 0$, let us define

$$I(s,\lambda;u) := s^3\lambda^4 \iint_{Q_T} e^{-2s\alpha}\xi^3|u|^2 dxdt + s\lambda^2 \iint_{Q_T} e^{-2s\alpha}\xi|\nabla u|^2 dxdt, \tag{2.8}$$

where

$$\alpha(t,x) := \frac{\exp(12\lambda\|\eta^0\|_\infty) - \exp[\lambda(10\|\eta^0\|_\infty - \eta^0(x))]}{t^5(T-t)^5} \quad \text{and} \quad \xi(t,x) := \frac{\exp[\lambda(10\|\eta^0\|_\infty - \eta^0(x))]}{t^5(T-t)^5}. \tag{2.9}$$

Here, $\eta^0 \in \mathcal{C}^2(\overline{\Omega})$ is a function satisfying

$$|\nabla\eta^0| \geqslant \kappa \text{ in } \Omega\backslash\omega_2, \quad \eta^0 > 0 \text{ in } \Omega \quad \text{and} \quad \eta^0 = 0 \text{ on } \partial\Omega,$$

with $\kappa > 0$. The proof of the existence of such a function η^0 can be found in [18, Lemma 1.1, Chap. 1] (see also [12, Lemma 2.68, Chap. 2]). We will use the two notations

$$\alpha^*(t) := \max_{x\in\overline{\Omega}} \alpha(t,x) \quad \text{and} \quad \xi^*(t) := \min_{x\in\overline{\Omega}} \xi(t,x), \tag{2.10}$$

for all $t \in (0,T)$.

2.2.1 Some auxiliary results

Let us now give some useful auxiliary results that we will need in our proofs. The first one is a Carleman estimate which holds for solutions of the heat equation with non-homogeneous Neumann boundary conditions:

Lemma 2.1. *Let us assume that $d > 0$, $u^0 \in L^2(\Omega)$, $f_1 \in L^2(Q_T)$ and $f_2 \in L^2(\Sigma_T)$. Then there exists a constant $C := C(\Omega,\omega_2) > 0$ such that the solution to the system*

$$\begin{cases} -\partial_t u - d\Delta u = f_1 & \text{in } Q_T, \\ \frac{\partial u}{\partial n} = f_2 & \text{on } \Sigma_T, \\ u(T,\cdot) = u^0 & \text{in } \Omega, \end{cases}$$

satisfies

$$I(s,\lambda;u) \leqslant C\left(s^3\lambda^4 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^3|u|^2 dxdt + \iint_{Q_T} e^{-2s\alpha}|f_1|^2 dxdt \right.$$
$$\left. + s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^*|f_2|^2 d\sigma dt\right),$$

for all $\lambda \geqslant C$ and $s \geqslant C(T^5 + T^{10})$.

The proof of Lemma 2.1 can essentially be found in [17]. In fact, in this article, the weights are a little bit different ($t(T-t)$ instead of $t^5(T-t)^5$) but the proof just needs to be slightly adapted to obtain the present result.

From Lemma 2.1, one can deduce the following result:

Lemma 2.2. *Let $h \in L^2(\Sigma_T)^m$. Then there exists a constant $C := C(\Omega, \omega_2) > 0$ such that for every $\varphi^0 \in L^2(\Omega)^m$, the solution φ to the system*

$$\begin{cases} -\partial_t \varphi = D\Delta\varphi - G^* \cdot \nabla\varphi + A^*\varphi & \text{in } Q_T, \\ \frac{\partial\varphi}{\partial n} = h & \text{on } \Sigma_T, \\ \varphi(T, \cdot) = \varphi^0 & \text{in } \Omega \end{cases} \tag{2.11}$$

satisfies

$$I(s, \lambda; \varphi) \leqslant C \left(s^3\lambda^4 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^3|\varphi|^2 dxdt + s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^* \left|h\right|^2 d\sigma dt \right),$$

for every $\lambda \geqslant C$ and $s \geqslant s_0 = C(T^5 + T^{10})$.

Proof of Lemma 2.2.

For $i \in \{1, ..., m\}$, we will apply first the Carleman inequality of Lemma 2.1 to the i^{th} equation of System (2.11) with $f_1 := \sum_{j=1}^m \{-g_{ji} \cdot \nabla\varphi_j + a_{ji}\varphi_j\}$ and $f_2 := \frac{\partial\varphi_i}{\partial n}$. Then we have, for every $s \geqslant C(T^5 + T^{10})$,

$$\begin{aligned} I(s, \lambda; \varphi_i) \leqslant C \Bigg(& s^3\lambda^4 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^3|\varphi_i|^2 dxdt \\ & + \iint_{Q_T} e^{-2s\alpha} \left| \sum_{j=1}^m \{-g_{ji} \cdot \nabla\varphi_j + a_{ji}\varphi_j\} \right|^2 dxdt + s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^* \left|\frac{\partial\varphi_i}{\partial n}\right|^2 d\sigma dt \Bigg). \end{aligned}$$

The sum of the expressions above for $i \in \{1, ..., m\}$ leads to the estimate

$$\begin{aligned} I(s, \lambda; \varphi) \leqslant C \Bigg(& s^3\lambda^4 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^3|\varphi|^2 dxdt \\ & + \iint_{Q_T} e^{-2s\alpha}\{|\nabla\varphi|^2 + |\varphi|^2\} dxdt + s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^* \left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma dt \Bigg). \end{aligned}$$

Going back to the definition of $I(s, \lambda; \varphi)$ (see (2.8)), we observe that one can absorb the global terms in $\nabla\varphi$ and φ of the right-hand side by taking s large enough (more precisely $s \geqslant C(T^5 + T^{10})$ with C large enough), which concludes the proof.

■

In this section, we will use also the following estimate.

Lemma 2.3. *Let $r \in \mathbb{R}$. Then there exists $C := C(r, \omega_2, \Omega) > 0$ such that, for every $T > 0$ and every $u \in L^2(0, T; H^1(\Omega))$,*

$$\begin{aligned} s^{r+2}\lambda^{r+2} \iint_{Q_T} e^{-2s\alpha}\xi^{r+2}|u|^2 dxdt \leqslant C \Bigg(& s^r\lambda^r \iint_{Q_T} e^{-2s\alpha}\xi^r|\nabla u|^2 dxdt \\ & + s^{r+2}\lambda^{r+2} \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^{r+2}|u|^2 dxdt \Bigg), \end{aligned}$$

for every $\lambda \geqslant C$ and $s \geqslant C(T^5 + T^{10})$.

The proof of this lemma can be found for example in [13, Lemma 3].

Our next Lemma is some Poincaré-type inequality involving the differential operator $\mathcal{N}^*$.

Lemma 2.4. *There exists a constant $C := C(\Omega) > 0$ such that for every $u \in H_0^1(\Omega)$, the following estimate holds:*

$$\int_\Omega u^2 \leqslant C \int_\Omega |\mathcal{N}^* u|^2, \tag{2.12}$$

where $\mathcal{N}^* := g_{mi_0} \cdot \nabla - a_{mi_0}$.

Proof of Lemma 2.4

Using Condition (2.1), the proof is obvious if $g_{mi_0} = 0$. Then, let us assume from now on that $g_{mi_0} \neq 0$. Let us treat first the particular case $g_{mi_0} = e_1$ with e_1 the first element of the canonical basis of $\mathbb{R}^N$ and consider the set

$$\Omega^\perp := \{\widehat{x}_1 := (x_2, ..., x_N) \in \mathbb{R}^{N-1} : \exists\ x_1 \in \mathbb{R}\ s.t.\ (x_1, \widehat{x}_1) \in \Omega\} \subset \mathbb{R}^{N-1}.$$

We can decompose the term in the right-hand side of inequality (2.12) as follows:

$$\begin{array}{rcl}\displaystyle\int_\Omega |\mathcal{N}^* u|^2 dx &=& \displaystyle\int_{\Omega^\perp} \int_{\{x_1 \in \mathbb{R} : (x_1, \widehat{x}_1) \in \Omega\}} (\partial_{x_1}(u) - a_{mi_0} u)^2 dx_1 d\widehat{x}_1 \\ &=& \displaystyle\int_{\Omega^\perp} \int_{\{x_1 \in \mathbb{R} : (x_1, \widehat{x}_1) \in \Omega\}} \{|\partial_{x_1}(u)|^2 - 2a_{mi_0} \partial_{x_1}(u) u + a_{mi_0}^2 u^2\} dx_1 \widehat{x}_1.\end{array}$$

We remark that for every $\widehat{x}_1 \in \Omega^\perp$

$$\int_{\{x_1 \in \mathbb{R} : (x_1, \widehat{x}_1) \in \Omega\}} 2a_{mi_0} \partial_{x_1}(u) u dx_1 = a_{mi_0} \int_{\{x_1 \in \mathbb{R} : (x_1, \widehat{x}_1) \in \Omega\}} \partial_{x_1}(u^2) dx_1 = 0.$$

Moreover, using the usual Poincaré inequality, for every $\widehat{x}_1 \in \Omega^\perp$,

$$\int_{\{x_1 \in \mathbb{R} : (x_1, \widehat{x}_1) \in \Omega\}} u^2 dx_1 \leqslant \kappa |\Omega|^2 \int_{\{x_1 \in \mathbb{R} : (x_1, \widehat{x}_1) \in \Omega\}} |\partial_{x_1}(u)|^2 dx_1,$$

where κ does not depend on Ω. Thus for every $\widehat{x}_1 \in \Omega^\perp$

$$\int_{\{x_1 \in \mathbb{R} : (x_1, \widehat{x}_1) \in \Omega\}} (\partial_{x_1}(u) - a_{mi_0} u)^2 dx_1 \quad \geqslant \quad (a_{mi_0}^2 + \kappa^{-1} |\Omega|^{-2}) \int_{\{x_1 \in \mathbb{R} : (x_1, \widehat{x}_1) \in \Omega\}} u^2 dx_1.$$

We conclude by integration on $\Omega^\perp$ of the last inequality.

To treat the general case $g_{mi_0} \neq e_1$, we just use an appropriate change of coordinates in which g_{mi_0} can be seen as e_1 and we apply exactly the same reasoning.

■

In order to deal with more regular solutions, one needs the following lemma:

Lemma 2.5. *Let $z_0 \in H_0^1(\Omega)^m$ and $f \in L^2(Q_T)^m$. Let us denote by $\mathcal{R} := -D\Delta - G \cdot \nabla - A$ and consider z the solution in $W_2^{2,1}(Q_T)$ to the system*

$$\left\{\begin{array}{ll} \partial_t z = D\Delta z + G \cdot \nabla z + Az + f & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ z(0, \cdot) = z_0 & \text{in } \Omega. \end{array}\right. \tag{2.13}$$

Let $d \in \mathbb{N}$. Let us assume that $z_0 \in H^{2d+1}(\Omega)^m$, $f \in L^2(0, T; H^{2d}(\Omega)^m) \cap H^d(0, T; L^2(\Omega)^m)$ and satisfy the following compatibility conditions:

$$\left\{\begin{array}{l} g_0 := z_0 \in H_0^1(\Omega)^m, \\ g_1 := f(0, \cdot) - \mathcal{R} g_0 \in H_0^1(\Omega)^m, \\ \dots \\ g_d := \partial_t^{d-1} f(0, \cdot) - \mathcal{R} g_{d-1} \in H_0^1(\Omega)^m. \end{array}\right.$$

Then $z \in L^2(0,T;H^{2d+2}(\Omega)^m) \cap H^{d+1}(0,T;L^2(\Omega)^m)$ *and we have the estimate*

$$\|z\|_{L^2(0,T;H^{2d+2}(\Omega)^m)\cap H^{d+1}(0,T;L^2(\Omega)^m)} \leqslant C(\|f\|_{L^2(0,T;H^{2d}(\Omega)^m)\cap H^d(0,T;L^2(\Omega)^m)} + \|z_0\|_{H^{2d+1}(\Omega)^m}). \tag{2.14}$$

It is a classical result that can be easily deduced for example from [16, Th. 6, p. 365].

2.2.2 Carleman inequality

We are now able to prove the following inequality:

Proposition 2.2. *There exists a constant* $C := C(\omega_0, \Omega) > 0$ *such that for every* $\psi^0 \in L^2(\Omega)^m$, *the corresponding solution* ψ *to System* (2.7) *satisfies*

$$\begin{aligned}\iint_{Q_T} e^{-2s\alpha}\{s^7\lambda^8\xi^7|\mathcal{N}^*\psi|^2 + s^5\lambda^6\xi^5|\nabla\mathcal{N}^*\psi|^2 + s^3\lambda^4\xi^3|\nabla\nabla\mathcal{N}^*\psi|^2 + s\lambda^2\xi|\nabla\nabla\nabla\mathcal{N}^*\psi|^2\}dxdt \\ \leqslant Cs^7\lambda^8\iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^7|\mathcal{N}^*\psi|^2dxdt,\end{aligned} \tag{2.15}$$

for every $\lambda \geqslant C$ *and* $s \geqslant s_0 = C(T^5 + T^{10})$.

Remark 2. It may be quite surprising that one can put so much derivatives at the left-hand-side of equality (2.15), because the initial condition ψ^0 is only supposed to be L^2, hence ψ is only assumed to be in $W(0,T)$ (see (1.3). However, because of the fact that the exponential weight $e^{-2s\alpha}$ is strong enough to absorb the singularity that only exists at initial time $t = 0$, it is quite easy to prove that all the integrals appearing in the left-hand side of (2.15) exist (this can notably be deduced for example from inequalities like (2.26), (2.27) or (2.28)).

Proof of Proposition 2.2.

The proof is inspired by [13]. The main difference here is that we keep $\mathcal{N}^*\psi$ at the right-hand side, which complicates a little bit the proof. Let us denote by

$$\mathcal{R} := -D\Delta + G^*\cdot\nabla - A^*. \tag{2.16}$$

We can assume without loss of generality that

$$\psi^0 \in H^5(\Omega) \text{ and } \psi^0, \mathcal{R}\psi^0, \mathcal{R}^2\psi^0 \in H^1_0(\Omega)^m$$

(The general case follows from a density argument). Thus, using Lemma 2.5, the solution ψ to System (2.7) is an element of $L^2(0,T;H^6(\Omega)^m) \cap H^3(0,T;L^2(\Omega)^m)$. First of all let us apply the differential operator

$$\nabla\nabla\mathcal{N}^* = \nabla\nabla(-a_{mi_0} + g_{mi_0}\cdot\nabla)$$

to System (2.7) satisfied by ψ. Thus, if we call $\phi := (\phi_{ij})_{1\leqslant i,j\leqslant N}$ with $\phi_{ij} := \partial_i\partial_j\mathcal{N}^*\psi$, then one observes that ϕ is a solution of the following system:

$$\begin{cases} -\partial_t\phi_{ij} = D\Delta\phi_{ij} - G^*\cdot\nabla\phi_{ij} + A^*\phi_{ij} & \text{in } Q_T, \\ \frac{\partial\phi}{\partial n} = \frac{\partial(\nabla\nabla\mathcal{N}^*\psi)}{\partial n} & \text{on } \Sigma_T, \\ \phi(T,\cdot) = \nabla\nabla\mathcal{N}^*\psi^0 & \text{in } \Omega. \end{cases} \tag{2.17}$$

By applying Lemma 2.2 to ϕ, we have

$$I(s,\lambda,\phi) \leqslant C\left(s^3\lambda^4\iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^3|\phi|^2dxdt + s\lambda\iint_{\Sigma_T} e^{-2s\alpha^*}\xi^*\left|\frac{\partial(\nabla\nabla\mathcal{N}^*\psi)}{\partial n}\right|^2 d\sigma dt\right), \tag{2.18}$$

for every $\lambda \geqslant C$ and $s \geqslant C(T^5 + T^{10})$.

The proof will be divided into three steps :

- In the first step, we will estimate the boundary term in the right-hand side of inequality (2.18) with some global interior term involving ψ that will be absorbed later.
- In the second step, we will compare $I(s,\lambda,\phi)$ with the left-hand side of inequality (2.15).
- Finally, in the last step, we will estimate the local term of high order appearing in inequality (2.18) thanks to some local terms that will be absorbed in the left-hand side of inequality (2.18) and also thanks to the local term of the right-hand side of inequality (2.15).

Step 1: Let us consider a function $\theta \in \mathcal{C}^2(\overline{\Omega})$ such that

$$\frac{\partial\theta}{\partial n} = \theta = 1 \text{ on } \partial\Omega.$$

After an integration by parts of the boundary term, we obtain

$$\begin{aligned}
&s\lambda\int_0^T e^{-2s\alpha^*}\xi^*\int_{\partial\Omega}\left|\frac{\partial\phi}{\partial n}\right|^2 d\sigma dt\\
&= s\lambda\int_0^T e^{-2s\alpha^*}\xi^*\int_{\partial\Omega}\left|\frac{\partial\phi}{\partial n}\right|\nabla\phi\cdot\nabla\theta d\sigma dt\\
&= s\lambda\int_0^T e^{-2s\alpha^*}\xi^*\int_{\Omega}\Delta\phi\nabla\phi\cdot\nabla\theta d\sigma dt + s\lambda\int_0^T e^{-2s\alpha^*}\xi^*\int_{\Omega}\nabla(\nabla\theta\cdot\nabla\phi)\cdot\nabla\phi d\sigma dt\\
&\qquad +\tfrac{s\lambda}{2}\int_0^T e^{-2s\alpha^*}\xi^*\int_{\Omega}\nabla\left|\nabla\phi\right|^2\cdot\nabla\theta d\sigma dt.
\end{aligned}$$

Using successively Cauchy-Schwarz inequality and Young's inequality, we deduce that

$$\begin{aligned}
s\lambda\int_0^T e^{-2s\alpha^*}\xi^*\int_{\partial\Omega}\left|\frac{\partial\phi}{\partial n}\right|^2 d\sigma dt &\leqslant C\lambda\int_0^T e^{-2s\alpha^*}\|(s\xi^*)^{4/5}\psi\|_{H^4(\Omega)^m}\|(s\xi^*)^{1/5}\psi\|_{H^5(\Omega)^m}dt\\
&\leqslant C\lambda\int_0^T e^{-2s\alpha^*}\|(s\xi^*)^{4/5}\psi\|^2_{H^4(\Omega)^m}dt\\
&\qquad +C\lambda\int_0^T e^{-2s\alpha^*}\|(s\xi^*)^{1/5}\psi\|^2_{H^5(\Omega)^m}dt.
\end{aligned} \tag{2.19}$$

Let us introduce $\widehat{\psi} := \rho\psi$ with $\rho \in \mathcal{C}^\infty([0,T])$ to be chosen later satisfying $\rho(0) = \partial_t\rho(0) = \partial_{tt}\rho(0) = \partial_{ttt}\rho(0) = 0$. Then $\widehat{\psi}$ is solution to the system

$$\begin{cases}
-\partial_t\widehat{\psi} = D\Delta\widehat{\psi} - G^*\cdot\nabla\widehat{\psi} + A^*\widehat{\psi} - \rho_t\psi & \text{in } Q_T,\\
\widehat{\psi} = 0 & \text{on } \Sigma_T,\\
\widehat{\psi}(T,\cdot) = 0 & \text{in } \Omega.
\end{cases} \tag{2.20}$$

Lemma 2.5 gives for $\widehat{\psi}$ the estimate

$$\|\widehat{\psi}\|_{L^2(0,T;H^{2d+2}(\Omega)^m)\cap H^{d+1}(0,T;L^2(\Omega)^m)} \leqslant C\|\rho_t\psi\|_{L^2(0,T;H^{2d}(\Omega)^m)\cap H^d(0,T;L^2(\Omega)^m)}, \tag{2.21}$$

for $d \in \{0,1,2\}$. Using the definitions of ξ^* and α^* given in (2.10), for $\rho := (s\xi^*)^a e^{-s\alpha^*}$ and $a \in \mathbb{R}^+$ we have

$$|\partial_t\rho| \leqslant CT(s\xi^*)^{a+6/5}e^{-s\alpha^*}, \tag{2.22}$$

$$|\partial_{tt}\rho| \leqslant CT^2(s\xi^*)^{a+12/5}e^{-s\alpha^*} \tag{2.23}$$

and

$$|\partial_{ttt}\rho| \leqslant CT^3(s\xi^*)^{a+18/5}e^{-s\alpha^*}. \tag{2.24}$$

Using inequality (2.21) with $\rho := e^{-2s\alpha^*}(s\xi^*)^{4/5}$ and $d = 1$, we obtain

$$\begin{aligned}&\int_0^T e^{-2s\alpha^*}(s\xi^*)^{4/5}\|\psi\|^2_{H^4(\Omega)^m}dt\\ \leqslant C&\left(\int_0^T \|\partial_t(e^{-2s\alpha^*}(s\xi^*)^{4/5})\psi\|^2_{H^2(\Omega)^m}dt + \int_0^T \|\partial_t(\partial_t(e^{-2s\alpha^*}(s\xi^*)^{4/5})\psi)\|^2_{L^2(\Omega)^m}dt\right).\end{aligned} \tag{2.25}$$

Applying now inequality (2.21) with $\rho := \partial_t(e^{-2s\alpha^*}(s\xi^*)^{4/5})$ and $d = 0$, we get

$$\begin{aligned}\int_0^T \|\partial_t(e^{-2s\alpha^*}(s\xi^*)^{4/5})\psi\|^2_{H^2(\Omega)^m}dt + \int_0^T \|\partial_t(\partial_t(e^{-2s\alpha^*}(s\xi^*)^{4/5})\psi)\|^2_{L^2(\Omega)^m}dt&\\ \leqslant C\int_0^T \|\partial_{tt}(e^{-2s\alpha^*}(s\xi^*)^{4/5})\psi\|^2_{L^2(\Omega)^m}dt.&\end{aligned} \tag{2.26}$$

Using (2.23) with $a = 4/5$ together with (2.26) and (2.25), we deduce

$$\int_0^T e^{-2s\alpha^*}\|(s\xi^*)^{4/5}\psi\|^2_{H^4(\Omega)^m}dt \leqslant CT^2\int_0^T e^{-2s\alpha^*}\|(s\xi^*)^{16/5}\psi\|^2_{L^2(\Omega)^m}dt. \tag{2.27}$$

Using exactly the same proof and taking into account (2.24), one can also prove that

$$\int_0^T e^{-2s\alpha^*}\|(s\xi^*)^{-2/5}\psi\|^2_{H^6(\Omega)^m}dt \leqslant CT^3\int_0^T e^{-2s\alpha^*}\|(s\xi^*)^{16/5}\psi\|^2_{L^2(\Omega)^m}dt. \tag{2.28}$$

Using inequalities (2.27)- (2.28) and the interpolation inequality

$$\|u\|_{H^5(\Omega)^m} \leqslant C\|u\|^{1/2}_{H^4(\Omega)^m}\|u\|^{1/2}_{H^6(\Omega)^m} \text{ for every } u \in H^6(\Omega)^m,$$

we deduce that

$$\begin{aligned}\int_0^T e^{-2s\alpha^*}\|(s\xi^*)^{1/5}\psi\|^2_{H^5(\Omega)^m}dt &\leqslant C\int_0^T e^{-2s\alpha^*}\|(s\xi^*)^{-2/5}\psi\|_{H^6(\Omega)^m}\|(s\xi^*)^{4/5}\psi\|_{H^4(\Omega)^m}dt\\ &\leqslant CT^{5/2}\int_0^T e^{-2s\alpha^*}\|(s\xi^*)^{16/5}\psi\|^2_{L^2(\Omega)^m}dt.\end{aligned} \tag{2.29}$$

Thus inequalities (2.18), (2.19), (2.27) and (2.29) lead to

$$\begin{aligned}I(s,\lambda;\phi) \leqslant C\Bigg(&s^3\lambda^4\iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^3|\phi|^2dxdt\\ &+\lambda s^{32/5}(T^2+T^{5/2})\int_{Q_T} e^{-2s\alpha^*}(\xi^*)^{32/5}|\psi|^2dxdt\Bigg),\end{aligned}$$

for every $s \geqslant C(T^5 + T^{10})$ and $\lambda \geqslant C$. Hence, since

$$T^2 + T^{5/2} \leqslant Cs^{2/5},$$

we have

$$I(s,\lambda;\phi) \leqslant C\left(s^3\lambda^4\iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^3|\phi|^2dxdt + \lambda s^{34/5}\int_{Q_T} e^{-2s\alpha^*}(\xi^*)^{34/5}|\psi|^2dxdt\right), \tag{2.30}$$

for every $s \geqslant C(T^5+T^{10})$ and $\lambda \geqslant C$.

Step 2: We apply Lemma 2.3 successively to $\mathcal{N}^*\psi$ with $r=5$, then to $\nabla\mathcal{N}^*\psi$ with $r=3$, and we obtain

$$\begin{aligned} s^7\lambda^8 \iint_{Q_T} e^{-2s\alpha}\xi^7|\mathcal{N}^*\psi|^2 dxdt \leqslant C \Bigg(& s^5\lambda^6 \iint_{Q_T} e^{-2s\alpha}\xi^5|\nabla\mathcal{N}^*\psi|^2 dxdt \\ & + s^7\lambda^8 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^7|\mathcal{N}^*\psi|^2 dxdt \Bigg) \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} s^5\lambda^6 \iint_{Q_T} e^{-2s\alpha}\xi^5|\nabla\mathcal{N}^*\psi|^2 dxdt \leqslant C \Bigg(& s^3\lambda^4 \iint_{Q_T} e^{-2s\alpha}\xi^3|\nabla\nabla\mathcal{N}^*\psi|^2 dxdt \\ & + s^5\lambda^6 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^5|\nabla\mathcal{N}^*\psi|^2 dxdt \Bigg), \end{aligned} \tag{2.32}$$

for every $\lambda \geqslant C$ and $s \geqslant C(T^5+T^{10})$. A combination of inequalities (2.30)-(2.32) gives

$$\begin{aligned} & \iint_{Q_T} e^{-2s\alpha}\{s^7\lambda^8\xi^7|\mathcal{N}^*\psi|^2 + s^5\lambda^6\xi^5|\nabla\mathcal{N}^*\psi|^2 + s^3\lambda^4\xi^3|\nabla\nabla\mathcal{N}^*\psi|^2 + s\lambda^2\xi|\nabla\nabla\nabla\mathcal{N}^*\psi|^2\} dxdt \\ & \leqslant C \Bigg(\lambda s^{34/5} \int_{Q_T} e^{-2s\alpha^*}(\xi^*)^{34/5}|\psi|^2 dxdt \\ & \qquad + \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\{s^7\lambda^8\xi^7|\mathcal{N}^*\psi|^2 + s^5\lambda^6\xi^5|\nabla\mathcal{N}^*\psi|^2 + s^3\lambda^4\xi^3|\nabla\nabla\mathcal{N}^*\psi|^2\} dxdt \Bigg). \end{aligned} \tag{2.33}$$

Step 3: Let us consider $\theta_1 \in \mathcal{C}^2(\overline{\Omega})$ such that

$$\begin{cases} \mathrm{supp}(\theta_1) \subseteq \omega_1, & \\ \theta_1 \equiv 1 & \text{in } \omega_2, \\ 0 \leqslant \theta_1 \leqslant 1 & \text{in } \Omega. \end{cases}$$

Then, after an integration by parts,

$$\begin{aligned} & s^3\lambda^4 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^3|\nabla\nabla\mathcal{N}^*\psi|^2 dxdt \\ & \leqslant s^3\lambda^4 \iint_{(0,T)\times\omega_1} \theta_1 e^{-2s\alpha}\xi^3|\nabla\nabla\mathcal{N}^*\psi|^2 dxdt \\ & = -s^3\lambda^4 \iint_{(0,T)\times\omega_1} \sum_{i,j=1}^N \{\partial_i(\theta_1 e^{-2s\alpha}\xi^3)\partial_i\partial_j\mathcal{N}^*\psi + \theta_1 e^{-2s\alpha}\xi^3\partial_i^2\partial_j\mathcal{N}^*\psi\}\partial_j(\mathcal{N}^*\psi) dxdt \\ & \leqslant Cs^3\lambda^4 \iint_{(0,T)\times\omega_1} \{|\nabla(\theta_1 e^{-2s\alpha}\xi^3)||\nabla\nabla\mathcal{N}^*\psi||\nabla\mathcal{N}^*\psi| + \theta_1 e^{-2s\alpha}\xi^3|\nabla\nabla\nabla\mathcal{N}^*\psi||\nabla\mathcal{N}^*\psi|\} dxdt. \end{aligned} \tag{2.34}$$

Using the definition of ξ and α given in (2.9), we deduce that

$$|\nabla(\theta_1 e^{-2s\alpha}\xi^3)| \leqslant Cs\lambda e^{-2s\alpha}\xi^4, \tag{2.35}$$

which, combined with Young's inequality, leads, for every $\varepsilon > 0$, to

$$\begin{aligned} & s^3\lambda^4 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^3|\nabla\nabla\mathcal{N}^*\psi|^2 dxdt \\ & \leqslant C \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\{\varepsilon s^3\lambda^4\xi^3|\nabla\nabla\mathcal{N}^*\psi|^2 + \varepsilon s\lambda^2\xi|\nabla\nabla\nabla\mathcal{N}^*\psi|^2 + C_\varepsilon s^5\lambda^6\xi^5|\nabla\mathcal{N}^*\psi|^2\} dxdt, \end{aligned} \tag{2.36}$$

where C_ε depends only on ε. Thus, thanks to inequalities (2.33) and (2.34), one can absorb (by taking ε small enough) the local terms involving $|\nabla\nabla\mathcal{N}^*\psi|^2$ and $|\nabla\nabla\nabla\mathcal{N}^*\psi|^2$ into the right-hand side of inequality (2.36) and obtain

$$\begin{aligned}
&\iint_{Q_T} e^{-2s\alpha}\{s^7\lambda^8\xi^7|\mathcal{N}^*\psi|^2 + s^5\lambda^6\xi^5|\nabla\mathcal{N}^*\psi|^2 + s^3\lambda^4\xi^3|\nabla\nabla\mathcal{N}^*\psi|^2 + s\lambda^2\xi|\nabla\nabla\nabla\mathcal{N}^*\psi|^2\}dxdt \\
&\leqslant C\left(\lambda s^{34/5}\int_{Q_T} e^{-2s\alpha^*}(\xi^*)^{34/5}|\psi|^2dxdt\right. \\
&\qquad\qquad \left. + \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\{s^7\lambda^8\xi^7|\mathcal{N}^*\psi|^2 + s^5\lambda^6\xi^5|\nabla\mathcal{N}^*\psi|^2\}dxdt\right).
\end{aligned} \tag{2.37}$$

Again, let us consider $\theta_0 \in \mathcal{C}^2(\overline{\Omega})$ such that

$$\begin{cases} \operatorname{supp}(\theta_0) \subseteq \omega_0, & \\ \theta_0 \equiv 1 & \text{in } \omega_1, \\ 0 \leqslant \theta_0 \leqslant 1 & \text{in } \Omega. \end{cases}$$

Then, after an integration by parts,

$$\begin{aligned}
&s^5\lambda^6\iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^5|\nabla\mathcal{N}^*\psi|^2dxdt \\
&\leqslant s^5\lambda^6\iint_{(0,T)\times\omega_0} \theta_0 e^{-2s\alpha}\xi^5|\nabla\mathcal{N}^*\psi|^2dxdt \\
&= -s^5\lambda^6\iint_{(0,T)\times\omega_0}\sum_{i=1}^N\{\partial_i(\theta_0 e^{-2s\alpha}\xi^5)\partial_i\mathcal{N}^*\psi + \theta_0 e^{-2s\alpha}\xi^5\partial_i^2\mathcal{N}^*\psi\}\mathcal{N}^*\psi dxdt \\
&\leqslant Cs^5\lambda^6\iint_{(0,T)\times\omega_0}\{|\nabla(\theta_0 e^{-2s\alpha}\xi^5)||\nabla\mathcal{N}^*\psi||\mathcal{N}^*\psi| + \theta_0 e^{-2s\alpha}\xi^5|\nabla\nabla\mathcal{N}^*\psi||\mathcal{N}^*\psi|\}dxdt.
\end{aligned}$$

Similarly to inequality (2.35), we have

$$|\nabla(\theta_0 e^{-2s\alpha}\xi^5)| \leqslant Cs\lambda e^{-2s\alpha}\xi^6,$$

which, combined with Young's inequality, leads, for every $\varepsilon > 0$, to

$$\begin{aligned}
&s^5\lambda^6\iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^5|\nabla\mathcal{N}^*\psi|^2dxdt \\
&\leqslant \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\{\varepsilon s^5\lambda^6\xi^5|\nabla\mathcal{N}^*\psi|^2 + \varepsilon s^3\lambda^4\xi^3|\nabla\nabla\mathcal{N}^*\psi|^2 + C_\varepsilon s^7\lambda^8\xi^7|\mathcal{N}^*\psi|^2\}dxdt,
\end{aligned} \tag{2.38}$$

where C_ε depends only on ε. Thus, thanks to inequalities (2.37) and (2.38), one can absorb (by taking ε small enough) the local terms involving $|\nabla\mathcal{N}^*\psi|^2$ and $|\nabla\nabla\mathcal{N}^*\psi|^2$ in the right-hand side and obtain

$$\begin{aligned}
&\iint_{Q_T} e^{-2s\alpha}\{s^7\lambda^8\xi^7|\mathcal{N}^*\psi|^2 + s^5\lambda^6\xi^5|\nabla\mathcal{N}^*\psi|^2 + s^3\lambda^4\xi^3|\nabla\nabla\mathcal{N}^*\psi|^2 + s\lambda^2\xi|\nabla\nabla\nabla\mathcal{N}^*\psi|^2\}dxdt \\
&\qquad \leqslant C\left(\lambda s^{34/5}\int_{Q_T} e^{-2s\alpha^*}(\xi^*)^{34/5}|\psi|^2dxdt + s^7\lambda^8\iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^7|\mathcal{N}^*\psi|^2dxdt\right).
\end{aligned} \tag{2.39}$$

Applying now Lemma 2.4, and using the definitions of α^* and ξ^* given in (2.10), we obtain the following inequalities:

$$\begin{aligned}
s^7\lambda^8\iint_{Q_T}(\xi^*)^7e^{-2s\alpha^*}|\psi|^2dxdt &\leqslant Cs^7\lambda^8\iint_{Q_T}(\xi^*)^7e^{-2s\alpha^*}|\mathcal{N}^*\psi|^2dxdt \\
&\leqslant Cs^7\lambda^8\iint_{Q_T}\xi^7e^{-2s\alpha}|\mathcal{N}^*\psi|^2dxdt.
\end{aligned} \tag{2.40}$$

The two last inequalities (2.39) and (2.40) give

$$\begin{aligned}
&s^7\lambda^8 \iint_{Q_T} (\xi^*)^7 e^{-2s\alpha^*}|\psi|^2 dxdt \\
&\quad + \iint_{Q_T} e^{-2s\alpha}\{s^7\lambda^8\xi^7|\mathcal{N}^*\psi|^2 + s^5\lambda^6\xi^5|\nabla\mathcal{N}^*\psi|^2 + s^3\lambda^4\xi^3|\nabla\nabla\mathcal{N}^*\psi|^2 + s\lambda^2\xi|\nabla\nabla\nabla\mathcal{N}^*\psi|^2\}dxdt \\
&\quad\quad \leqslant C\left(\lambda s^{34/5}\int_{Q_T} e^{-2s\alpha^*}(\xi^*)^{34/5}|\psi|^2 dxdt + s^7\lambda^8\iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^7|\mathcal{N}^*\psi|^2 dxdt\right).
\end{aligned}$$

Hence, since $34/5 < 7$, one can absorb the global term of the right-hand side by taking s large enough and obtain inequality (2.15).

■

Thanks to our Carleman inequality, we can deduce the following observability inequality:

Proposition 2.3. *Then for every $\psi^0 \in L^2(\Omega)^m$, the solution ψ in $W(0,T)^m$ to System* (2.7) *satisfies*

$$\int_\Omega |\psi(0,x)|^2 dx \leqslant C_{obs}\iint_{(0,T)\times\omega_0} e^{-2s_0\alpha}\xi^7|\mathcal{N}^*\psi|^2 dxdt, \tag{2.41}$$

where $C_{obs} := Ce^{C(1+T+1/T^5)}$.

The proof of Proposition 2.3 is very classical and is mainly based on dissipation estimates and the fact that the weights are bounded from below by some positive constant as soon as we are far from 0 and T (for example on $(T/4, 3T/4)$, together with the fact that Lemma 2.4 leads to the inequality

$$\iint_{(T/4,3T/4)\times\Omega}|\psi|^2 dxdt \quad\leqslant\quad C\iint_{(T/4,3T/4)\times\Omega}|\mathcal{N}^*\psi|^2 dxdt.$$

2.3 Analytic resolution

This section is devoted to constructing a solution to the analytic control problem (1.16), with a control regular enough belonging to the range of the differential operator $\mathcal{N}$. We recall that the definition of $\mathcal{N}$ is given in (2.2). Let us consider $\theta \in \mathcal{C}^2(\overline{\Omega})$ such that

$$\begin{cases} \mathrm{supp}(\theta) \subseteq \omega, & \\ \theta \equiv 1 & \text{in } \omega_0, \\ 0 \leqslant \theta \leqslant 1 & \text{in } \Omega. \end{cases} \tag{2.42}$$

Proposition 2.4. *Let us assume that Condition* (2.1) *holds. Consider the system*

$$\begin{cases} \partial_t z = D\Delta z + G\cdot\nabla z + Az + \mathcal{N}(\theta v) & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ z(0,\cdot) = y^0 & \text{in } \Omega. \end{cases} \tag{2.43}$$

Then System (2.43) *is null controllable at time T, i.e. for every $y^0 \in L^2(\Omega)^m$, there exists a control $v \in L^2(Q_T)^m$ such that the solution z to System* (2.43) *satisfies $z(T) \equiv 0$ in Ω. Moreover, for every $K \in (0,1)$ we have $e^{Ks_0\alpha^*}v \in W_2^{2,1}(Q_T)^m$ (the definition of $W_2^{2,1}(Q_T)$ is given in* (1.4)*) and*

$$\|e^{Ks_0\alpha^*}v\|_{W_2^{2,1}(Q_T)^m} \leqslant e^{C(1+T+1/T^5)}\|y^0\|_{L^2(\Omega)^m}. \tag{2.44}$$

Proof of Proposition 2.4.
We will use the usual duality method developed by Fursikov and Imanuvilov in [18] in the spirit of what was done in [9] to obtain more regular controls. Let $y^0 \in L^2(\Omega)^m$ and ρ be the weight defined by

$$\rho := \xi^7 e^{-2s_0\alpha}.$$

Let $k \in \mathbb{N}^*$ and let us consider the following optimal control problem

$$\begin{cases} \text{minimize } J_k(v) := \dfrac{1}{2}\displaystyle\int_{Q_T} \rho^{-1}|v|^2 dxdt + \dfrac{k}{2}\int_\Omega |z(T)|^2 dx, \\ v \in L^2(Q_T, \rho^{-1/2})^m, \end{cases} \tag{2.45}$$

where z is the solution in $W(0,T)^m$ to System (2.43).

The functional $J_k : L^2(Q_T, \rho^{-1/2})^m \to \mathbb{R}^+$ is differentiable, coercive and strictly convex on the space $L^2(Q_T, \rho^{-1/2})^m$. Therefore there exists a unique solution to the control optimal problem (2.45) (see [24, p. 128]) and the optimal control v_k is characterized thanks to the solution z_k of the primal system

$$\begin{cases} \partial_t z_k = D\Delta z_k + G\cdot\nabla z_k + Az_k + \mathcal{N}(\theta v_k) & \text{in } Q_T, \\ z_k = 0 & \text{on } \Sigma_T, \\ z_k(0,\cdot) = y^0 & \text{in } \Omega, \end{cases} \tag{2.46}$$

the solution φ_k to the dual system

$$\begin{cases} -\partial_t \varphi_k = D\Delta\varphi_k - G^*\cdot\nabla\varphi_k + A^*\varphi_k & \text{in } Q_T, \\ \varphi_k = 0 & \text{on } \Sigma_T, \\ \varphi_k(T,\cdot) = kz_k(T,\cdot) & \text{in } \Omega \end{cases} \tag{2.47}$$

and the relation

$$\begin{cases} v_k = -\rho\theta\mathcal{N}^*\varphi_k \text{ in } Q_T, \\ v_k \in L^2(Q_T, \rho^{-1/2})^m. \end{cases} \tag{2.48}$$

The rest of the proof is divided into two steps. In the first step, we will prove that the sequence $(v_k)_{k\in\mathbb{N}^*}$ converges to a control $v \in L^2(Q_T, \rho^{-1/2})^m$ with an associated solution z to System (2.43) satisfying $z(T) \equiv 0$ in Ω. Then, in the second step, we will establish (2.44).

Step 1:
Firstly, the characterization (2.46), (2.47) and (2.48) of the minimizer v_k of J_k in $L^2(Q_T, \rho^{-1/2})^m$ leads to the following computations

$$\begin{array}{rcl} J_k(v_k) & = & -\dfrac{1}{2}\displaystyle\int_0^T \langle \theta\mathcal{N}^*\varphi_k, v_k\rangle_{L^2(\Omega)^m} dt + \dfrac{1}{2}\langle z_k(T), \varphi_k(T)\rangle_{L^2(\Omega)^m} \\ & = & -\dfrac{1}{2}\displaystyle\int_0^T \langle \varphi_k, \mathcal{N}(\theta v_k)\rangle_{L^2(\Omega)^m} dt + \dfrac{1}{2}\int_0^T \{\langle z_k, \partial_t\varphi_k\rangle_{L^2(\Omega)^m} + \langle \partial_t z_k, \varphi_k\rangle_{L^2(\Omega)^m}\} dt \\ & & \qquad\qquad + \dfrac{1}{2}\langle y^0, \varphi_k(0,\cdot)\rangle_{L^2(\Omega)^m} \\ & = & \dfrac{1}{2}\langle y^0, \varphi_k(0,\cdot)\rangle_{L^2(\Omega)^m}. \end{array}$$

Then, for every $\varepsilon > 0$,

$$J_k(v_k) \leqslant \frac{\varepsilon}{4}\|\varphi_k(0,\cdot)\|^2_{L^2(\Omega)^m} + \frac{1}{4\varepsilon}\|y^0\|^2_{L^2(\Omega)^m}. \tag{2.49}$$

Moreover, using the definition of J_k, the definition of θ and our observability inequality (2.41),

$$\|\varphi_k(0,\cdot)\|^2_{L^2(\Omega)^m} \leqslant C_{obs}\iint_{Q_T} \rho\theta^2|\mathcal{N}^*\varphi_k|^2 dxdt = C_{obs}\iint_{Q_T} \rho^{-1}|v_k|^2 dxdt \leqslant 2C_{obs}J_k(v_k). \tag{2.50}$$

Then, using inequalities (2.49) and (2.50),

$$J_k(v_k) \leqslant \frac{\varepsilon}{2}C_{obs}J_k(v_k) + \frac{1}{4\varepsilon}\|y^0\|^2_{L^2(\Omega)^m}.$$

Thus, for $\varepsilon := C_{obs}^{-1}$,

$$J_k(v_k) \leqslant \frac{C_{obs}}{2}\|y^0\|^2_{L^2(\Omega)^m}. \tag{2.51}$$

Furthermore, we have (see [24])

$$\begin{aligned}\|z_k\|_{W(0,T)^m} &\leqslant C\left(\|\mathcal{N}(\theta v_k)\|_{L^2(0,T;H^{-1}(\Omega))^m} + \|y^0\|_{L^2(\Omega)^m}\right),\\ &\leqslant C\left(\|v_k\|_{L^2(Q_T)} + \|y^0\|_{L^2(\Omega)^m}\right),\\ &\leqslant C(1+C_{obs})\|y^0\|_{L^2(\Omega)^m},\end{aligned} \tag{2.52}$$

where C does not depend on y^0, T and k. Then, using inequalities (2.51) and (2.52), we deduce that there exist subsequences, which are still denoted v_k, z_k, such that the following weak convergences hold:

$$\begin{cases} v_k \rightharpoonup v & \text{in } L^2(Q_T,\rho^{-1/2})^m,\\ z_k \rightharpoonup z & \text{in } W(0,T)^m,\\ z_k(T) \rightharpoonup 0 & \text{in } L^2(\Omega)^m.\end{cases}$$

Passing to the limit in k, z is solution to System (2.43). As $W(0,T)^m$ is compactly included in $\mathcal{C}([0,T];L^2(\Omega)^m)$ (see for example [15, p. 570]) then $z_k \to z$ strongly in $\mathcal{C}([0,T];L^2(\Omega)^m)$ and $z(T) \equiv 0$ in Ω. Thus the solution z to System (2.43) with control $v \in L^2(Q_T,\rho^{-1/2})^m$ satisfies $z(T) \equiv 0$ in Ω and using (2.51), we obtain the inequality

$$\|v\|^2_{L^2(Q_T,\rho^{-1/2})^m} \leqslant C_{obs}\|y^0\|^2_{L^2(\Omega)^m}. \tag{2.53}$$

Step 2: One remarks that for every $K \in (0,1)$, there exists a constant $C > 0$ such that $e^{2Ks_0\alpha^*} \leqslant C\xi^{-7}e^{2s_0\alpha}$. This inequality and estimate (2.51) imply that

$$\|e^{Ks_0\alpha^*}v_k\|^2_{L^2(Q_T)^m} \leqslant C\int_{Q_T}\xi^{-7}e^{2s_0\alpha}|v_k|^2dxdt \leqslant J(v_k) \leqslant e^{C(1+T+1/T^5)}\|y^0\|^2_{L^2(\Omega)^m}. \tag{2.54}$$

We recall that v_k is defined in (2.48), moreover, thanks to the definitions of ξ and α given in (2.9), one has, for every $\eta \geqslant 0$,

$$\begin{aligned}|\nabla(\xi^\eta e^{-2s_0\alpha})| &\leqslant C\xi^{\eta+1}e^{-2s_0\alpha},\\ |\Delta(\xi^\eta e^{-2s_0\alpha})| &\leqslant C\xi^{\eta+2}e^{-2s_0\alpha},\\ |\partial_t(\xi^\eta e^{-2s_0\alpha})| &\leqslant CT\xi^{\eta+6/5}e^{-2s_0\alpha}.\end{aligned}$$

These above inequalities, for $\eta := 7$, lead to the fact that

$$\|e^{Ks_0\alpha^*}\nabla v_k\|^2_{L^2(Q_T)^m} \leqslant C\iint_{Q_T} e^{-4s_0\alpha+2Ks_0\alpha^*}\{\xi^{14}|\nabla\mathcal{N}^*\varphi_k|^2 + \xi^{16}|\mathcal{N}^*\varphi_k|^2\}dxdt, \tag{2.55}$$

$$\|e^{Ks_0\alpha^*}\Delta v_k\|^2_{L^2(Q_T)^m} \leqslant C\iint_{Q_T} e^{-4s_0\alpha+2Ks_0\alpha^*}\{\xi^{14}|\nabla\nabla\mathcal{N}^*\varphi_k|^2 + \xi^{16}|\nabla\mathcal{N}^*\varphi_k|^2 + \xi^{18}|\mathcal{N}^*\varphi_k|^2\}dxdt \tag{2.56}$$

and

$$\begin{aligned}\|\partial_t(e^{Ks_0\alpha^*}v_k)\|^2_{L^2(Q_T)^m} &\leqslant CT\iint_{Q_T} e^{-4s_0\alpha+2Ks_0\alpha^*}\{\xi^{14}|\mathcal{N}^*\partial_t\varphi_k|^2 + \xi^{82/5}|\mathcal{N}^*\varphi_k|^2\}dxdt\\ &\leqslant CT\iint_{Q_T} e^{-4s_0\alpha}\{\xi^{14}(|\mathcal{N}^*\varphi_k|^2 + |\Delta\mathcal{N}^*\varphi_k|^2 + |\nabla\mathcal{N}^*\varphi_k|^2) + \xi^{82/5}|\mathcal{N}^*\varphi_k|^2\}dxdt.\end{aligned} \tag{2.57}$$

For every $\eta,\nu > 0$ there exists a constant $C_{\eta,\nu}$ such that

$$|\xi^\eta e^{-4s\alpha+2Ks_0\alpha^*}| \leqslant C_{\eta,\nu}\xi^\nu e^{-2s\alpha}.$$

Combining the last inequality with (2.54)-(2.57), we deduce that

$$\begin{array}{c}\|e^{Ks_0\alpha^*}v_k\|^2_{W^{2,1}_2(Q_T)^m}\\ \leqslant e^{C(1+T+1/T^5)}\displaystyle\iint_{Q_T}e^{-2s_0\alpha}\{(s_0\xi)^7|\mathcal{N}^*\varphi_k|^2+(s_0\xi)^5|\nabla\mathcal{N}^*\varphi_k|^2+(s_0\xi)^3|\nabla\nabla\mathcal{N}^*\varphi_k|^2dxdt.\end{array}$$

Using (2.15), we obtain that $e^{Ks_0\alpha^*}\nabla v_k$, $e^{Ks_0\alpha^*}\Delta v_k$, $\partial_t(e^{Ks_0\alpha^*}v_k)\in L^2(Q_T)^m$, and that

$$\|e^{Ks_0\alpha^*}v_k\|^2_{W^{2,1}_2(Q_T)^m}\leqslant e^{C(1+T+1/T^5)}\iint_{Q_T}e^{-2s\alpha}(s_0\xi)^7|\theta\mathcal{N}^*\varphi_k|^2dxdt=e^{C(1+T+1/T^5)}\|v_k\|^2_{L^2(Q_T)^m}.$$

The estimate (2.51) of v_k gives

$$\|e^{Ks_0\alpha^*}v_k\|_{W^{2,1}_2(Q_T)^m}\leqslant e^{C(1+T+1/T^5)}\|y^0\|_{L^2(\Omega)^m}.$$

We conclude by letting $k\to+\infty$ (after extracting an adequate subsequence) in the equalities above.

■

2.4 End of the proof of Theorem 1

Let us assume that Condition 1.5 holds. We will prove first the null controllability at time T of System (1.1). Let $y^0\in L^2(\Omega)^m$. We follow the method explained in Section 1.3. Let us remind that θ is defined in (2.42). Using Proposition 2.4, the following system:

$$\left\{\begin{array}{ll}\partial_t z=D\Delta z+G\cdot\nabla z+Az+\mathcal{N}(\theta v) & \text{in } Q_T,\\ z=0 & \text{on } \Sigma_T,\\ z(0,\cdot)=y^0 & \text{in } \Omega\end{array}\right.\tag{2.58}$$

is null controllable at time T, thus there exists a control $v\in L^2(Q_T)^m$ such that the solution z in $W(0,T)^m$ to System (2.58) satisfies

$$z(T)\equiv 0 \text{ in } \Omega.$$

Moreover

$$e^{Ks_0\alpha^*}v\in W^{2,1}_2(Q_T)^m.\tag{2.59}$$

Taking into account Proposition 2.1 and the definition of $\mathcal{M}^*$ given in (2.6), one has (2.3) with the operator

$$\begin{array}{cccc}\mathcal{M}: & W^{2,1}_2(Q_T)^m & \to & L^2(Q_T)^m\times L^2(Q_T)^{m-1}\\ & f & \mapsto & \mathcal{M}f,\end{array}$$

defined by

$$\mathcal{M}f=\begin{pmatrix}0\\ \vdots\\ 0\\ f_m \quad (i_0^{th}\text{ line})\\ 0\\ \vdots\\ 0\\ -(g_{mi_0}\cdot\nabla+a_{mi_0})f_1+(g_{1i_0}\cdot\nabla+a_{1i_0})f_m\\ \vdots\\ -(g_{mi_0}\cdot\nabla+a_{mi_0})f_{i_0-1}+(g_{(i_0-1)i_0}\cdot\nabla+a_{(i_0-1)i_0})f_m\\ (-\partial_t+d_{i_0}\Delta)f_m-(g_{mi_0}\cdot\nabla+a_{mi_0})f_{i_0}+(g_{i_0i_0}\cdot\nabla+a_{i_0i_0})f_m\\ -(g_{mi_0}\cdot\nabla+a_{mi_0})f_{i_0-1}+(g_{(i_0-1)i_0}\cdot\nabla+a_{(i_0-1)i_0})f_m\\ \vdots\\ -(g_{mi_0}\cdot\nabla+a_{mi_0})f_{m-1}+(g_{(m-1)i_0}\cdot\nabla+a_{(m-1)i_0})f_m\end{pmatrix}.$$

Let $(\widehat{z},\widehat{v})$ be defined by

$$\begin{pmatrix} \widehat{z} \\ \widehat{v} \end{pmatrix} := \mathcal{M}\,(\theta v)\,.$$

Using (2.59) and the fact that $\mathcal{M}$ is a differential operator of order 1 in time and 2 in space (see Proposition 2.1) with bounded coefficients, we obtain that $(\widehat{z},\widehat{v}) \in L^2(Q_T)^m \times L^2(Q_T)^{m-1}$. Moreover, using (2.59) we have $\widehat{z}(0,\cdot) = \widehat{z}(T,\cdot) = 0$ in Ω and we remark that $(\widehat{z},\widehat{v})$ is a solution to the control problem

$$\begin{cases} \partial_t \widehat{z} = D\Delta\widehat{z} + G\cdot\nabla\widehat{z} + A\widehat{z} + B\widehat{v} + \mathcal{N}(\theta v) & \text{in } Q_T, \\ \widehat{z} = 0 & \text{on } \Sigma_T, \\ \widehat{z}(0,\cdot) = \widehat{z}(T,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{2.60}$$

in particular $\mathcal{L}\circ\mathcal{M} = \mathcal{N}$. Finally, $(\widehat{z},\widehat{v}) \in W(0,T)^m \times L^2(Q_T)^{m-1}$ thanks to the usual parabolic regularity. Thus the pair $(y,u) := (z-\widehat{z}, -\widehat{v})$ is a solution to System (1.1) in $W(0,T)^m \times L^2(Q_T)^{m-1}$ and satisfies

$$y(T,\cdot) \equiv 0 \ \text{ in } \ \Omega.$$

Let us assume now that Condition (1.5) does not hold, and let us prove that the approximate controllability at time T of System (2.43) does not hold. Let us consider the adjoint system

$$\begin{cases} -\partial_t\psi_1 = d_1\Delta\psi_1 - \sum\limits_{j=1}^{m-1} g_{j1}\cdot\nabla\psi_j + \sum\limits_{j=1}^{m-1} a_{j1}\psi_j & \text{in } Q_T, \\ -\partial_t\psi_2 = d_2\Delta\psi_2 - \sum\limits_{j=1}^{m-1} g_{j2}\cdot\nabla\psi_j + \sum\limits_{j=1}^{m-1} a_{j2}\psi_j & \text{in } Q_T, \\ \dots \\ -\partial_t\psi_{m-1} = d_{m-1}\Delta\psi_{m-1} - \sum\limits_{j=1}^{m-1} g_{j(m-1)}\cdot\nabla\psi_j + \sum\limits_{j=1}^{m-1} a_{j(m-1)}\psi_j & \text{in } Q_T, \\ -\partial_t\psi_m = d_m\Delta\psi_m - \sum\limits_{j=1}^{m} g_{jm}\cdot\nabla\psi_j + \sum\limits_{j=1}^{m} a_{jm}\psi_j & \text{in } Q_T, \\ \psi_1 = \ldots = \psi_m = 0 & \text{on } \Sigma_T, \\ \psi_1(T,\cdot) = \psi_1^0, \ \ldots, \psi_m(T,\cdot) = \psi_m^0 & \text{in } \Omega. \end{cases} \tag{2.61}$$

It is well-known (see [12, Th. 2.43, p. 56])) that the approximate controllability to System (2.43) is equivalent to the following unique continuation property for the solutions to System (2.61):

$$\psi_1 = \ldots = \psi_{m-1} \equiv 0 \text{ in } q_T \Rightarrow \psi \equiv 0 \text{ in } Q_T. \tag{2.62}$$

But for $\psi_1^0 = \ldots = \psi_{m-1}^0 \equiv 0$ in Ω and every $\varphi_m^0 \not\equiv 0$, Property (2.62) is not satisfied. Indeed, we remark first that in this case, we have $\psi_1 = \ldots = \psi_{m-1} \equiv 0$ in Q_T (and hence notably on q_T) but $\psi_m \not\equiv 0$ in Q_T. Thus System (2.43) is not approximately controllable at time T. This completes the proof of Theorem 1 since we have

Condition (1.5) $\Rightarrow$ Null Controllability $\Rightarrow$ Approximate Controllability $\Rightarrow$ Condition (1.5).

3 Proof of Theorem 2

Let us remind that in this case, we have only 2 equations and 1 space dimension.

3.1 Algebraic resolution

We will assume in this section that all differential operators are defined in $\mathcal{C}^\infty(Q_T)$. We recall that $\mathcal{N}$ is simply the identity operator and $\mathcal{L}$ is given in (1.20). We want to find a differential operator $\mathcal{M}$ satisfying

$$\mathcal{L}\circ\mathcal{M} = Id. \tag{3.1}$$

Let us emphasize that when coefficients are depending on time and space, we prove equality (1.21) with the identity operator in the right-hand side (and not $\mathcal{N}$ as defined in (2.2)), that is equality (3.1), because Proposition 2.4 holds only for constant coefficients and does not seem to be adapted to the case of nonconstant coefficients. We have the following proposition:

Proposition 3.1. *One has*

(i) Under Conditions (1.7) *and* (1.10)*, there exists a differential operator $\mathcal{M}$ of order at most* 1 *in time and* 2 *in space, with bounded coefficients on $(a,b)\times\mathcal{O}$, such that equality* (3.1) *holds.*

(ii) Under Condition (1.8) *and* (1.11)*, there exists a differential operator $\mathcal{M}$ of order at most* 2 *in time,* 4 *in space, and* $1-2$ *respectively in crossed space-time, with bounded coefficients on $(a,b)\times\mathcal{O}$, such that equality* (3.1) *holds.*

Proof of Proposition 3.1.
Equality (3.1) is equivalent to

$$\mathcal{M}^*\circ\mathcal{L}^* = Id. \tag{3.2}$$

Taking into account the definition of $\mathcal{L}$ given in (1.20), the adjoint $\mathcal{L}^*$ of the operator $\mathcal{L}$ is given by

$$\mathcal{L}^*\varphi := \begin{pmatrix} \mathcal{L}_1^*\varphi \\ \mathcal{L}_2^*\varphi \\ \mathcal{L}_3^*\varphi \end{pmatrix} = \begin{pmatrix} -\partial_t\varphi_1 - \partial_x(d_1\partial_x\varphi_1) + \partial_x(g_{11}\varphi_1) + \partial_x(g_{21}\varphi_2) - a_{11}\varphi_1 - a_{21}\varphi_2 \\ -\partial_t\varphi_2 - \partial_x(d_2\partial_x\varphi_2) + \partial_x(g_{12}\varphi_1) + \partial_x(g_{22}\varphi_2) - a_{12}\varphi_1 - a_{22}\varphi_2 \\ \varphi_1 \end{pmatrix}. \tag{3.3}$$

We remark first that

$$\mathcal{L}_1^*\varphi + \{\partial_t + \partial_x(d_1\partial_x\cdot) - \partial_x(g_{11}\cdot) + a_{11}\}\circ\mathcal{L}_3^*\varphi = \partial_x(g_{21}\varphi_2) - a_{21}\varphi_2. \tag{3.4}$$

(i) Since $g_{21}=0$ in $(a,b)\times\mathcal{O}$, one can just consider $\mathcal{M}^*$ defined for every $\psi := (\psi_1,\psi_2,\psi_3)\in \mathcal{C}^\infty(Q_T)^3$, locally on $(a,b)\times\mathcal{O}$ (where $a_{21}\neq 0$ thanks to Condition (1.7)), by

$$\mathcal{M}^*\psi := \begin{pmatrix} \psi_3 \\ -\frac{\psi_1 + \{\partial_t + \partial_x(d_1\partial_x\cdot) - \partial_x(g_{11}\cdot) + a_{11}\}\psi_3}{a_{21}} \end{pmatrix}, \tag{3.5}$$

so that equality (3.2) is satisfied. Moreover, the coefficients of $\mathcal{M}^*$ (and hence of $\mathcal{M}$) are bounded.

(ii) Under Condition (1.8), one can proceed as follows: we consider the operator

$$\mathcal{Q}(\varphi) = \begin{pmatrix} \mathcal{L}_3^*\varphi \\ \mathcal{L}_1^*\varphi + \{\partial_t + \partial_x(d_1\partial_x\cdot) - \partial_x(g_{11}\cdot) + a_{11}\}\circ\mathcal{L}_3^*\varphi \\ \partial_x(\mathcal{L}_1^*\varphi + \{\partial_t + \partial_x(d_1\partial_x\cdot) - \partial_x(g_{11}\cdot) + a_{11}\}\circ\mathcal{L}_3^*\varphi) \\ \partial_t(\mathcal{L}_1^*\varphi + \{\partial_t + \partial_x(d_1\partial_x\cdot) - \partial_x(g_{11}\cdot) + a_{11}\}\circ\mathcal{L}_3^*\varphi) \\ \partial_{xx}(\mathcal{L}_1^*\varphi + \{\partial_t + \partial_x(d_1\partial_x\cdot) - \partial_x(g_{11}\cdot) + a_{11}\}\circ\mathcal{L}_3^*\varphi) \\ \mathcal{L}_2^*\varphi + \{a_{12} - \partial_x(g_{12}\cdot)\}\circ\mathcal{L}_3^*\varphi, \\ \partial_x(\mathcal{L}_2^*\varphi + \{a_{12} - \partial_x(g_{12}\cdot)\}\circ\mathcal{L}_3^*\varphi) \end{pmatrix}, \tag{3.6}$$

i.e.

$$\mathcal{Q}(\varphi) = \begin{pmatrix} \varphi_1 \\ (-a_{21} + \partial_x g_{21})\varphi_2 + g_{21}\partial_x\varphi_2 \\ (-\partial_x a_{21} + \partial_{xx}g_{21})\varphi_2 + (-a_{21} + 2\partial_x g_{21})\partial_x\varphi_2 + g_{21}\partial_{xx}\varphi_2 \\ (-\partial_t a_{21} + \partial_{tx}g_{21})\varphi_2 + \partial_t g_{21}\partial_x\varphi_2 + (-a_{21} + \partial_x g_{21})\partial_t\varphi_2 + g_{21}\partial_{xt}\varphi_2 \\ (-\partial_{xx}a_{21} + \partial_{xxx}g_{21})\varphi_2 + (-2\partial_x a_{21} + 3\partial_{xx}g_{21})\partial_x\varphi_2 + (-a_{21} + 3\partial_x g_{21})\partial_{xx}\varphi_2 + g_{21}\partial_{xxx}\varphi_2 \\ (-a_{22} + \partial_x g_{22})\varphi_2 + (-\partial_x d_2 + g_{22})\partial_x\varphi_2 - \partial_t\varphi_2 - d_2\partial_{xx}\varphi_2 \\ (-\partial_x a_{22} + \partial_{xx}g_{22})\varphi_2 + (-\partial_{xx}d_2 - a_{22} + 2\partial_x g_{22})\partial_x\varphi_2 + (-2\partial_x d_2 + g_{22})\partial_{xx}\varphi_2 - \partial_{tx}\varphi_2 - d_2\partial_{xxx}\varphi_2 \end{pmatrix}. \tag{3.7}$$

It is easy to see that there are only 7 different derivatives of φ appearing in (3.6), which are the following ones:

$$\varphi_1, \varphi_2, \partial_x\varphi_2, \partial_t\varphi_2, \partial_{xx}\varphi_2, \partial_{xt}\varphi_2, \partial_{xxx}\varphi_2.$$

Hence, we can see the operator $\mathcal{Q}$ as a matrix M acting on these derivatives (see [14, Sec. 3.2]), so that M is a square matrix of size 7×7. More precisely, one has

$$\mathcal{Q}(\varphi) = M(\varphi_1, \varphi_2, \partial_x\varphi_2, \partial_t\varphi_2, \partial_{xx}\varphi_2, \partial_{xt}\varphi_2, \partial_{xxx}\varphi_2),$$

with

$$M := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_{21} + \partial_x g_{21} & g_{21} & 0 & 0 & 0 & 0 \\ 0 & -\partial_x a_{21} + \partial_{xx} g_{21} & -a_{21} + 2\partial_x g_{21} & 0 & g_{21} & 0 & 0 \\ 0 & -\partial_t a_{21} + \partial_{tx} g_{21} & \partial_t g_{21} & -a_{21} + \partial_x g_{21} & 0 & g_{21} & 0 \\ 0 & -\partial_{xx} a_{21} + \partial_{xxx} g_{21} & -2\partial_x a_{21} + 3\partial_{xx} g_{21} & 0 & -a_{21} + 3\partial_x g_{21} & 0 & g_{21} \\ 0 & -a_{22} + \partial_x g_{22} & -\partial_x d_2 + g_{22} & -1 & -d_2 & 0 & 0 \\ 0 & -\partial_x a_{22} + \partial_{xx} g_{22} & -\partial_{xx} d_2 - a_{22} + 2\partial_x g_{22} & 0 & -2\partial_x d_2 + g_{22} & -1 & -d_2 \end{pmatrix}. \tag{3.8}$$

Matrix M is invertible since Condition (1.8) is verified. Let us call P the projection on the two first components

$$P(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (x_1, x_2).$$

Then, by definition of the inverse, we have

$$PM^{-1}M(\varphi_1, \dots, \partial_{xxx}\varphi_2) = \varphi.$$

Using (3.6), we remark that the previous equality can be rewritten as

$$PM^{-1}\mathcal{Q}(\varphi) = PM^{-1}\mathcal{S} \circ \mathcal{L}^*\varphi = \varphi,$$

where

$$\mathcal{S} := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & \mathcal{R}_1 \\ \partial_x & 0 & \partial_x \circ \mathcal{R}_1 \\ \partial_t & 0 & \partial_t \circ \mathcal{R}_1 \\ \partial_{xx} & 0 & \partial_{xx} \circ \mathcal{R}_1 \\ 0 & 1 & \mathcal{R}_2 \\ 0 & \partial_x & \partial_x \circ \mathcal{R}_2 \end{pmatrix}, \tag{3.9}$$

with $\mathcal{R}_1 := \partial_t + \partial_x(d_1\partial_x \cdot) - \partial_x(g_{11}\cdot) + a_{11}$ and $\mathcal{R}_2 := a_{12} - \partial_x(g_{12}\cdot)$. Hence equality (3.2) is satisfied for

$$\mathcal{M}^* := PM^{-1}\mathcal{S}. \tag{3.10}$$

Moreover, thanks to Conditions (1.8), the coefficients of $\mathcal{M}^*$ (and hence of $\mathcal{M}$) are bounded.

■

3.2 Analytic resolution

Let ω_1 be a nonempty open subsets included in ω satisfying

$$\overline{\omega}_1 \subset \omega_0.$$

Let us consider $\theta \in \mathcal{C}^2(\overline{\Omega})$ such that

$$\begin{cases} \mathrm{supp}(\theta) \subseteq \omega_0, & \\ \theta \equiv 1 & \text{in } \omega_1, \\ 0 \leqslant \theta \leqslant 1 & \text{in } \Omega. \end{cases} \tag{3.11}$$

We are going to explain what are the main differences with Subsection 2.2 in order to obtain a Carleman inequality. First of all, we need to find an equivalent to Lemma 2.1, which is the following:

Lemma 3.1. *Let us assume that $u^0 \in L^2(\Omega)$, $f_1 \in L^2(Q_T)$ and $d \in W^1_\infty(Q_T)$ such that $d > C > 0$ in Q_T. Then there exists a constant $C := C(\Omega, \omega_2) > 0$ such that the solution to the system*

$$\begin{cases} -\partial_t u - \partial_x(d\partial_x u) = f_1 & \text{in } Q_T, \\ \frac{\partial u}{\partial n} = f_2 & \text{on } \Sigma_T, \\ u(T, \cdot) = u^0 & \text{in } \Omega, \end{cases}$$

satisfies

$$I(s, \lambda; u) \leqslant C\left(s^3\lambda^4 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^3|u|^2 dxdt + \iint_{Q_T} e^{-2s\alpha}|f_1|^2 dxdt \right.$$
$$\left. + s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^*|f_2|^2 d\sigma dt\right),$$

for all $\lambda \geqslant C$ and $s \geqslant C(T^5 + T^{10})$.

The proof of Lemma 2.1 can be easily obtained by using the method of [17] together with the Carleman estimate proved in [23]. Let us consider the backward system

$$\begin{cases} -\partial_t\psi_1 = \partial_x(d_1\partial_x\psi_1) - \partial_x(g_{11}\psi_1) - \partial_x(g_{21}\psi_2) + a_{11}\psi_1 + a_{21}\psi_2 & \text{in } Q_T, \\ -\partial_t\psi_2 = \partial_x(d_2\partial_x\psi_2) - \partial_x(g_{12}\psi_1) - \partial_x(g_{22}\psi_2) + a_{12}\psi_1 + a_{22}\psi_2 & \text{in } Q_T, \\ \psi_1 = \psi_2 = 0 & \text{on } \Sigma_T, \\ \psi_1(T, \cdot) = \psi_1^0,\ \psi_2(T, \cdot) = \psi_2^0 & \text{in } \Omega, \end{cases} \tag{3.12}$$

where $\psi^0 := (\psi_1^0, \psi_2^0) \in L^2(\Omega)^2$. From the last Lemma, one can deduce:

Proposition 3.2. *One has*

(i) Under Condition (1.7) *and* (1.10)*, there exists a constant $C := C(\omega_1, \Omega) > 0$ such that for every $\psi^0 := (\psi_1^0, \psi_2^0) \in L^2(\Omega)^2$, the corresponding solution $\psi := (\psi_1, \psi_2)$ of the backward problem* (3.12) *satisfies*

$$\begin{aligned} \iint_{Q_T} e^{-2s\alpha}\{&s^9\lambda^{10}\xi^9|\psi|^2 + s^7\lambda^8\xi^7|\nabla\psi|^2 + s^5\lambda^6\xi^5|\nabla\nabla\psi|^2 \\ &+ s^3\lambda^4\xi^3|\nabla\nabla\nabla\psi|^2 + s\lambda^2\xi|\nabla\nabla\nabla\nabla\psi|^2 \\ &\leqslant Cs^9\lambda^{10} \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^9|\psi|^2 dxdt. \end{aligned} \tag{3.13}$$

(ii) Under Condition (1.8) *and* (1.11)*, we obtain the same conclusion as in item (i) by replacing estimate* (3.13) *by*

$$\begin{aligned} \iint_{Q_T} e^{-2s\alpha}\{&s^5\lambda^6\xi^5|\psi|^2 + s^3\lambda^4\xi^3|\nabla\psi|^2 + s\lambda^2\xi|\nabla\nabla\psi|^2\}dxdt \\ &\leqslant Cs^5\lambda^6 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^5|\psi|^2 dxdt. \end{aligned}$$

The proof is very similar to the proof of Proposition 2.2, the only difference is the beginning of the proof, we apply the operator ∇ to the equation (3.12) in the case (1.7) and the operators $\nabla\nabla\nabla$ in the case (1.8). After that we exactly follow the steps 1,2, and 3 of the proof of Proposition 2.2.

As a consequence, we also can derive the following observability inequality, whose proof is very classical (see also Proposition 2.3):

Proposition 3.3. *Under assumptions* (1.7) *and* (1.10) *or under assumptions* (1.8) *and* (1.11), *for every* $\psi^0 \in L^2(\Omega)^2$, *the solution to System* (3.12) *satisfies*

$$\int_\Omega |\psi(0,x)|^2 dx \leqslant C_{obs} \iint_{(0,T)\times\omega_1} e^{-2s_0\alpha}\xi^9|\psi|^2 dxdt, \tag{3.14}$$

where $s_0 := C(T^5 + T^{10})$ *and* $C_{obs} := e^{C(1+T+1/T^5)}$.

To conclude, one can obtain the following controllability result:

Proposition 3.4. *Consider the following system:*

$$\begin{cases} \partial_t z_1 = \partial_x(d_1\partial_x z_1) + g_{11}\partial_x z_1 + g_{12}\partial_x z_2 + a_{11}z_1 + a_{12}z_2 + \theta v_1 & \text{in } Q_T, \\ \partial_t z_2 = \partial_x(d_2\partial_x z_2) + g_{21}\partial_x z_1 + g_{22}\partial_x z_2 + a_{21}z_1 + a_{22}z_2 + \theta v_2 & \text{in } Q_T, \\ z_1 = z_2 = 0 & \text{on } \Sigma_T, \\ z_1(0,\cdot) = y_1^0,\ z_2(0,\cdot) = y_2^0 & \text{in } \Omega. \end{cases} \tag{3.15}$$

Under Conditions (1.7) *and* (1.10) *or under Condition* (1.8) *and* (1.11), *System* (3.15) *is null controllable at time* T, *that is for every* $y^0 \in L^2(\Omega)^2$ *there exists a control* $v := (v_1, v_2) \in L^2(Q_T)^2$ *such that the solution* z *to System* (3.15) *satisfies* $z(T) \equiv 0$ *in* Ω. *Moreover foreveryK* $\in (0,1)$, *we have* $e^{Ks_0\alpha^*}v \in \mathcal{X}^2$ *where:*

(i) Under Conditions (1.7) *and* (1.11), $\mathcal{X}$ *is defined by*

$$\mathcal{X} := L^2(0,T; H^4(\Omega) \cap H_0^1(\Omega)) \cap H^2(0,T; L^2(\Omega)). \tag{3.16}$$

(ii) Under Conditions (1.8) *and* (1.11), $\mathcal{X}$ *is defined by*

$$\mathcal{X} := L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega)). \tag{3.17}$$

Moreover in the both cases, we have the estimate

$$\|e^{Ks_0\alpha^*}v\|_{\mathcal{X}^2} \leqslant e^{C(1+T+1/T^5)}\|y^0\|_{L^2(\Omega)^2}. \tag{3.18}$$

One more time, the proof is very similar to the proof of Proposition 2.3, notably and one can recover easily estimates on the derivatives of order 1 and 2 in time of the control by using equation (3.12) verified by φ and estimates similar to (2.57).

3.3 Proof of Theorem 2

The proof is totally similar to the one of Theorem 1. Let us assume that one of the two Conditions (1.7) or (1.8) holds, and let us prove that System (1.6) is null controllable at time T (which will imply the approximate controllability at time T). Let $y^0 \in L^2(\Omega)^2$. Using Proposition 3.4, System (3.15) is null controllable at time T, more precisely there exists a control $v \in L^2(Q_T)^2$ such that the solution z in $W(0,T)^2$ to System (3.15) satisfies

$$z(T) \equiv 0 \text{ in } \Omega.$$

Moreover

$$e^{Ks_0\alpha^*}v \in \mathcal{X}^2. \tag{3.19}$$

Let us remind that θ was defined in (3.11). Let $(\widehat{z}_1, \widehat{z}_2, \widehat{v})$ be defined by

$$\begin{pmatrix} \widehat{z}_1 \\ \widehat{z}_2 \\ \widehat{v} \end{pmatrix} := \mathcal{M}\begin{pmatrix} \theta v_1 \\ \theta v_2 \end{pmatrix},$$

with $\mathcal{M} : \mathcal{X}^2 \to L^2(Q_T)^2 \times L^2(Q_T)$ given as the adjoint of $\mathcal{M}^*$ defined in (3.5) and (3.10). Thanks to the definition of θ given in (3.11), the fact the coefficients of $\mathcal{M}$ are necessarily at least in $L^\infty((a,b)\times\omega_0)$, the definition of $\mathcal{X}$ given in (3.16)-(3.17) and the fact that $\mathcal{M}$ is of order 1 in time and 2 in space under Condition (1.7) and is of order 2 in time, 4 in space and $1-2$ in crossed time-space (which is an interpolation space between $L^2((0,T),H^4(0,L))$ and $H^2((0,T),L^2(0,L))$ thanks to [25, 13.2, P. 96]) under Condition (1.8), we obtain $(\widehat{z}_1,\widehat{z}_2,\widehat{v}) \in L^2(Q_T)^2 \times L^2(Q_T)$. Moreover, using (3.19), we remark that $(\widehat{z}_1,\widehat{z}_2,\widehat{v})$ is a solution in to the control problem

$$\begin{cases} \partial_t\widehat{z} = \partial_x(D\partial_x\widehat{z}) + G\partial_x\widehat{z} + A\widehat{z} + B\widehat{v} + \theta v & \text{in } Q_T,\\ \widehat{z} = 0 & \text{on } \Sigma_T,\\ \widehat{z}(0,\cdot) = \widehat{z}(T,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{3.20}$$

Finally, $\widehat{z} \in W(0,T)^2$ thanks to the usual parabolic regularity. Thus the pair $(y,u) := (z-\widehat{z}, -\widehat{v})$ is a solution to System (1.6) in $W(0,T)^2 \times L^2(Q_T)$ and satisfies

$$y(T) \equiv 0 \ \text{ in } \ \Omega,$$

which concludes the proof of Theorem 2.

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